Moduli of sheaves on K3 surfaces

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1 Symplectic structures on moduli of sheaves

In this section we will outline a more general construction of a symplectic structure on the moduli of sheaves. Our main references for this section are [Muk84] and [HL97]. In the following S will always denote a smooth projective surface over $\mathbb{C}$, even though we can work in a more general setting. We fix a polarization h on S.

Recall, a sheaf E on S is called simple if $Hom(E, E) \cong \mathbb{C}$. In particular, h-stable sheaves are simple.

Fact 1.1. The moduli functor of simple sheaves may not be representable by a scheme, but there exist a coarse moduli space Spl_S. Furthermore, the functor is represented by an algebraic space $\mathbf{Spl}_S$. In more concrete terms, for all morphisms $\phi : B \to \mathrm{Spl}_S$, there exists an etale cover $\pi : \tilde{B} \to B$ and a sheaf $\mathcal{E}$ on $\tilde{B} \times S$, which is flat over $\tilde{B}$, such that for each closed point $x \in \tilde{B}$, the composition $\phi \circ \pi$ maps x to the point corresponding to simple sheaf $\mathcal{E}|_{\{x\}\times S}$.

1.1 Smoothness of the moduli

In this section we will make some remarks about smoothness of Spl_S. We will skip details as smoothness was more or less proven in the case of K3(see Benjamin's lecture notes). Note in this case this easily follows from vanishing of $Ext^2(E, E)_0$. The main theorem proven in [Muk84] is:

Theorem 1.1. *Let E_0 be a simple sheaf on S. Then Spl_S is smooth at E_0 if:*

i Pic_S is smooth. Note this condition is immediate for S is K3 or characteristic is 0(hence in our case too).

ii The natural map $j_{E_0} : H^0(S, \omega_S) \to Hom(E_0, E_0 \otimes \omega_S)$ is surjective.

Remark 1.1. The dimension can be computed easily using Grothendieck-Riemann-Roch. We will do this later for K3 surfaces.

Remark 1.2. Smoothness roughly means that finite order infinitesimal deformations of E_0 can be extended to arbitrary order. Strictly speaking, Mukai proves formal smoothness, but smoothness follows from this and local structure of Spl_S.

To prove smoothness, we will start with a finite order infinitesimal deformation as mentioned above. Then, we will attempt to extend it to higher order on the affine pieces of S. This is possible, but for a global deformation, these local deformations should match in the intersections. The failure to match will give us (Cech) cohomology classes, the obstruction classes, and we will conclude that when it is a coboundary we can actually make corrections so that the local deformations match to make a global one. Then, we will proceed to sketch a proof that the obstruction class vanishes in cohomology, hence we will be done.

Before giving the obstruction classes let's make some assumptions which doesn't effect generality. First, if E_0 is sky-scraper, then Spl_S is locally isomorphic to S at E_0, hence it is smooth. Also, twisting by ample line bundle gives automorphisms of Spl_S, so we can assume E_0 is generated by global sections and has vanishing higher cohomology. In this case we have a short exact sequence:

$$*_0 : 0 \to G_0 \to \mathcal{O}_S^n \to E_0 \to 0$$

Moreover, it can be shown that G_0 is locally free(see [Muk84]). We will try to deform short exact sequences as above.

Now, let $(A, \mathfrak{m})$ be an Artin local ring over $\mathbb{C}$ and $I \subset A$ be an ideal such that $I\mathfrak{m} = 0$ (for instance $A = \mathbb{C}[\epsilon]/(\epsilon^{n+1})$ and $I = (\epsilon^n)$). Let $(\overline{A}, \overline{\mathfrak{m}}) = (A/I, \mathfrak{m}/I)$ and let $\overline{E}$ be a deformation of E_0 over $\overline{A}$, i.e. a flat family of sheaves in Spl_S over $Spec(\overline{A})$, which restricts to E_0 over the central fiber $Spec(\overline{A}/\overline{m}) = Spec(\mathbb{C})$. Assume $\overline{E}$ lies in a short exact sequence as above, i.e.

$$\overline{*} : 0 \to \overline{G} \to \mathcal{O}_{S_{\overline{A}}}^n \to \overline{E} \to 0$$

which extends $*_0$ to $Spec(\overline{A})$.

We can define two obstruction classes: $ob(\overline{G}) \in Ext^2_S(G_0, G_0 \otimes_{\mathbb{C}} I)$, resp. $ob(\overline{*}) \in Ext^1_S(G_0, E_0 \otimes_{\mathbb{C}} I)$, which vanish if and only if $\overline{G}$, resp. $\overline{*}$ deform to G, resp. $* : 0 \to G \to \mathcal{O}_{S_A}^n \to E \to 0$, over S_A. Thus, vanishing of $ob(\overline{*})$ proves that $\overline{E}$ also deforms and finishes the proof of smoothness.

Mukai defines them by diagram chasing and states that they can be produced by deforming $\overline{G}$ and $\overline{*}$ on affine open subsets. We now outline this for $ob(\overline{G})$.

Choose an affine open cover $\{U_i\}$ of S. Then, $\overline{G}|_{U_i}$ and $\overline{G}|_{U_j}$ are equal(isomorphic by the identity map) when restricted to $U_i \cap U_j$. We can deform the vector bundle $\overline{G}|_{U_i}$ on $(U_i)|_{\overline{A}}$ to a vector bundle G_i on $(U_i)_A$ and extend the isomorphism(identity) to $g_{ij} : G_i|_{U_i \cap U_j} \xrightarrow{\cong} G_j|_{U_i \cap U_j}$, so that $(g_{ij})_{\overline{A}} = id$. Now to make a vector bundle out of G_i we need the cocycle condition:

$$g_{jk} \circ g_{ij} = g_{ik}$$

This is a non-linear condition, but we already now that it holds over $\overline{A}$. Hence, this relation holds modulo an element d_{ijk} of $Hom_{\mathcal{O}_{S_A}}(G_i|_{U_{ijk}}, G_k \otimes_A I|_{U_{ijk}})$, where $U_{ijk} = U_i \cap U_j \cap U_k$. Note that this is isomorphic to

$$Hom_{\mathcal{O}_S}((G_0)|_{U_{ijk}}, (G_0) \otimes_{\mathbb{C}} I|_{U_{ijk}})$$

Hence, the difference defines a 2-cocyle $\{d_{ijk}\}$. As G_0 is locally free, a 2-cocycles with values in $Hom_{\mathcal{O}_S}((G_0)|_{U_{ijk}}, (G_0) \otimes_{\mathbb{C}} I|_{U_{ijk}})$ gives an element

$$ob(\overline{G}) \in Ext^2_S(G_0, G_0 \otimes_{\mathbb{C}} I)$$

Clearly, it vanishes in cohomology if and only if $\overline{G}$ deforms to a vector bundle G over S_A.

One can play a similar game and try to deform $\overline{*}$ on affine pieces. This time differences gives a 1-cocycle, hence a class

$$ob(\overline{*}) \in Ext^1_S(G_0, E_0 \otimes_{\mathbb{C}} I)$$

It is slightly trickier in this case, but not much harder. Now let us state some properties of these obstruction classes. For the proofs and details see [Muk84].

1. Let $x_0 : E_0 \to G_0$ be the extension class of $*_0$. Then the composition

   $$G_0 \xrightarrow{ob(\overline{*})} E_0 \otimes_{\mathbb{C}} I[1] \xrightarrow{x_0 \otimes_{\mathbb{C}} 1_I} G_0 \otimes_{\mathbb{C}} I[2]$$

   is equal to $ob(\overline{G})$. Note, it is easy to see such a relation should hold simply because extendability of $\overline{*}$ implies extendability of $\overline{G}$ (hence if $ob(\overline{*})$ vanishes, then $ob(\overline{G})$ vanishes).

2. $ob(det\overline{G}) \in Ext^2_S(detG_0, detG_0 \otimes_{\mathbb{C}} I) \cong H^2(S, \mathcal{O}_S) \otimes_{\mathbb{C}} I$ is equal to $tr(ob\overline{G})$, where $tr : Ext^2_S(G_0, G_0 \otimes_{\mathbb{C}} I) \to H^2(S, \mathcal{O}_S) \otimes_{\mathbb{C}} I$ (see [HL97] if you want to see a definition). Note this is where the condition on the smoothness of $Pic(S)$ is used; namely $ob(det\overline{G}) = 0$ always.

Above remarks imply that $tr \circ (x_0 \otimes 1)$ sends $ob(\overline{*})$ to $ob(det\overline{G}) = 0$. Hence, it suffices to show that the composition

$$Ext^1(G_0, E_0) \xrightarrow{x_0 \circ} Ext^2(G_0, G_0) \xrightarrow{tr} H^2(S, \mathcal{O}_S)$$

is injective. But using Serre duality we see that injectivity of the composition above is equivalent to surjectivity of composition below

$$Ext^1(E_0, G_0 \otimes \omega_S) \leftarrow Hom(G_0, G_0 \otimes \omega_S) \leftarrow H^0(\omega_S)$$

, where the first arrow is still the Yoneda map with respect to x_0 and the second map is the most natural one that you can imagine. We have the commutative square

$$\begin{tikzcd} H^0(\omega_S) \arrow[r] \arrow[d] & Hom(G_0, G_0 \otimes \omega_S) \arrow[d] \\ Hom(E_0, E_0 \otimes \omega_S) \arrow[r] & Ext^1(E_0, G_0 \otimes \omega_S) \end{tikzcd}$$

But we already know the surjectivity of the left arrow in the diagram follows from the second assumption of the theorem, which is immediate for K3 and surjectivity of the bottom arrow is a result of a long exact sequence argument(for details see [Muk84]). Hence, we have smoothness.

1.2 The bilinear form

We have seen that $T_{[E_0]}\mathrm{Spl}_S \cong Ext^1(E_0, E_0)$ (where the isomorphism can be described explicitly as the Kodaira-Spencer map, which I will tell later). But the classes in $Ext^n(E_0, E_0)$ can naturally be seen as morphisms $E_0 \to E_0[n]$ in the derived category. Hence, we have a composition map

$$Ext^1(E_0, E_0) \times Ext^1(E_0, E_0) \to Ext^2(E_0, E_0)$$

where $(\alpha, \beta) \mapsto \alpha \circ \beta$. Using the trace map $E_0^\vee \overset{L}{\otimes} E_0 \to \mathcal{O}_S$ we obtain

$$Ext^2(E_0, E_0) \xrightarrow{tr} H^2(\mathcal{O}_S)$$

which we again call trace map(actually we have already used this above). Serre duality tells us $H^2(\mathcal{O}_S) \cong H^0(\omega_S)^\vee$ and choice of a global holomorphic 2-form $\alpha \in H^0(\omega_S)$ gives us a map $H^2(\mathcal{O}_S) \cong H^0(\omega_S)^\vee \xrightarrow{\alpha} \mathbb{C}$. So by composing Kodaira-Spencer map, $\circ$, tr and α, we get a bilinear map

$$T_{[E_0]}\mathrm{Spl}_S \times T_{[E_0]}\mathrm{Spl}_S \to \mathbb{C}$$

This is the pointwise description of the symplectic form on Spl_S. To show that this is "continuous" we have to show this can be done in families. Before that note in the case of a K3 surface choice of α is unique upto scaling but sometimes it can effect the form that we get and even its non-degeneracy(so we would have Poisson structures).

Now, let B be a smooth variety and $\mathcal{E}$ be a family of simple sheaves over S paramerized by B (i.e. a sheaf over $B \times S$, flat over B, simple over its closed points). Let $R\mathcal{H}om_B = R\pi_{B*}R\mathcal{H}om$, where π_B is the projection $B \times S \to B$ and let $\mathcal{E}xt^i_B$ denote its cohomology sheaves. These are sheaves over B and intuitively homs of families of sheaves parametrized by B. The Yoneda pairing exists in this case too:

$$\mathcal{E}xt^1_B(\mathcal{E}, \mathcal{E}) \times \mathcal{E}xt^1_B(\mathcal{E}, \mathcal{E}) \to \mathcal{E}xt^2_B(\mathcal{E}, \mathcal{E})$$

Similarly trace map extends to the family

$$\mathcal{E}xt^2(\mathcal{E},\mathcal{E}) \to R^2\pi_{B*}\mathcal{O}_{B\times S} = H^2(\mathcal{O}_S)\otimes_{\mathbb{C}}\mathcal{O}_B$$

Hence, choice of α as above gives a sheaf homomorphism

$$\mathcal{E}xt^1_B(\mathcal{E},\mathcal{E})\otimes_{\mathcal{O}_B}\mathcal{E}xt^1_B(\mathcal{E},\mathcal{E}) \to \mathcal{O}_B$$

We only need to set the isomorphism between tangent space and the ext group over families.

Aside. ***Atiyah class and Kodaira-Spencer map*** *For a given smooth complex variety X we have a short exact sequence*

$$0 \to \eta_*\Omega^1_X \to \mathcal{O}_{X\times X}/\mathcal{I}^2_\Delta \to \mathcal{O}_\Delta \to 0$$

where $\eta : X \to X\times X$ is the diagonal embedding, Δ the diagonal and $\mathcal{I}_\Delta$ its ideal sheaf. Note this holds as $\eta_*\Omega^1_X \cong \mathcal{I}_\Delta/\mathcal{I}^2_\Delta$. Hence, we get an extension class $\mathcal{O}_\Delta \to \eta_*\Omega^1_X[1]$ and if we denote the projection to first and second X by p and q, then we get a natural transformation

$$id_{D^b(X)}(\ ) \simeq Rp_*(q^*(\ )\otimes\mathcal{O}_\Delta) \to Rp_*(q^*(\ )\otimes\eta_*\Omega^1_X[1]) \simeq (\ )\otimes_{\mathcal{O}_X}\Omega^1_X[1]$$

hence a class $A(\mathcal{F}) \in Ext^1_X(\mathcal{F},\mathcal{F}\otimes\Omega^1_X)$. This is called the Atiyah class. It can be described in terms of complex differential geometry as well. Note $tr(exp(A(\mathcal{F})) = ch(\mathcal{F})$.

Now if $\mathcal{E} \to B\times S$ as above, then as $\Omega^1_{B\times S} \cong \pi^*_B\Omega^1_B \oplus \pi^*_S\Omega^1_S$, $A(\mathcal{E})$ decomposes into $A(\mathcal{E})' \in Ext^1(\mathcal{E},\mathcal{E}\otimes\pi^*_B\Omega^1_B)$ and $A(\mathcal{E})'' \in Ext^1(\mathcal{E},\mathcal{E}\otimes\pi^*_S\Omega^1_S)$. It is then clear that $A(\mathcal{E})'$ gives a sheaf homomorphism:

$$TB \xrightarrow{KS} \mathcal{E}xt^1_B(\mathcal{E},\mathcal{E})$$

We call this Kodaira-Spencer map. When the classifying map $B \to Spl_S$ is etale, this is an isomorphism, by the pointwise result.

Hence, composing the Kodaira-Spencer map with the above bilinear form over B we get a sheaf homomorphism

$$TB\otimes_{\mathcal{O}_B} TB \to \mathcal{O}_B$$

Now, as $\mathbf{Spl}_S$ is represented by an algebraic space $\mathbf{Spl}_S$, there exist an etale cover $\pi : B \to \mathrm{Spl}_S$ with a corresponding sheaf $\mathcal{E} \to B\times S$. Above remarks tell us how to define the form on TB, but we already have the map at a fiberwise level on Spl_S hence it descends to a bilinear form

$$T\mathrm{Spl}_S \otimes_{\mathcal{O}_{\mathrm{Spl}_S}} T\mathrm{Spl}_S \to \mathcal{O}_{\mathrm{Spl}_S}$$

. This is the form we want to show to be symplectic.

1.3 Symplecticity

Alternating:

The idea is very simple. In general composition is not symmetric or alternating. But for vector spaces we have the identity $tr(A \circ B) = tr(B \circ A)$, which turns into $tr(A \circ B) = (-1)^{|A||B|} tr(B \circ A)$ for super(or graded) vector spaces and graded operators A and B. The same holds for the trace map here and as we compose odd degree maps we get the alternating property. For more details see [Muk84] or [HL97].

Non-degeneracy:

For this I assume $\omega_S \cong \mathcal{O}_S$. Recall that for an n-dimensional smooth proper variety X the Serre duality pairing is given by

$$Ext^{n-i}(F, E \otimes \omega_X) \times Ext^i(E, F) \xrightarrow{tr \circ Yon} H^n(\omega_X) \cong H^0(\mathcal{O}_X)^\vee \cong \mathbb{C}$$

Hence, contraction of this with a global 2-form gives our bilinear pairing. But Serre duality says this map is non-degenrate, hence our alternating form is non-degenerate under the assumption that $\omega_S \cong \mathcal{O}_S$ (and $\alpha \neq 0$).

Closedness

In the case most relevant to us the dimension of the moduli will be 2, hence this will be immediate. But, we still want to outline an argument and refer to [HL97] for more details.

We will use upper prime $'$ (as in $A(\mathcal{E})'$) to refer to relevant part of a class when it has a decomposition similar to that of Atiyah class of $\mathcal{E} \to B \times S$.

Now as above $A(\mathcal{E})' \in Ext^1(\mathcal{E}, \mathcal{E} \otimes \pi_B^* \Omega_B^1)$ gives rise to $TB \to \mathcal{E}xt_B^1(\mathcal{E}, \mathcal{E})$. Hence taking squares

$$(A(\mathcal{E})^2)' \in Ext^2(\mathcal{E}, \mathcal{E} \otimes \pi_B^* \Omega_B^2) \text{ gives } \Lambda^2 TB \to \mathcal{E}xt_B^2(\mathcal{E}, \mathcal{E})$$

in the same way(note to obtain a morphism out of such a map, we need to use flat base change formula, but we prefer to stay sloppy here). Taking traces, and taking the appropriate relevant parts, we can show that

$$tr(A(\mathcal{E})^2)' \in H^2(\mathcal{O}_S) \otimes H^0(\Omega_B^2)$$

can be used to obtain a map

$$\Lambda^2 TB \to R^2 \pi_{B*} \mathcal{O}_{B \times S} \cong H^2(\mathcal{O}_S) \otimes \mathcal{O}_B$$

Choice of an element $H^0(\omega_S)$ and contraction with the last line gives us our bilinear pairing again. The reason we gave this description was to show that in order to prove closedness, it is enough to prove that $tr(A(\mathcal{F})^k)$ is closed for any $k \in \mathbb{N}$ and any sheaf $\mathcal{F}$ on a smooth projective variety X. One still needs to check that De Rham differential is compatible with the relevant parts mentioned above.

To prove $tr(A(\mathcal{F})^k)$ is closed, we can assume $\mathcal{F}$ is a line bundle by splitting principle. Then, choose an trivializing affine cover $\{U_i\}$ and transition functions $g_{ij} \in \mathcal{O}^*$ for $\mathcal{F}$. Then Atiyah class is the same as first Chern class and is given by the 1-cocyle $g_{ij}^{-1}dg_{ij} = "d(log(g_{ij}))"$. This is closed with respect to algebraic de Rham differential, finishing the sketch of the proof.

For other perspectives and generalizations see [Bot95] and [KM09].

2 Case of K3 surfaces

In this section, we will mainly follow [Muk87]. We specify to the case where S is a K3 surface. Also, h will always denote a polarization on S.

Definition 2.1. Let E be a coherent sheaf or more generally an element of $D^b(S) := D^bCoh(S)$. Then, the **Mukai vector**, $v(E)$ is defined to be $\sqrt{td(S)}ch(E) \in H^*(S,\mathbb{Q})$.

When S is a K3 surface $\sqrt{td(S)} = 1+w_S = (1,0,1)$ and $v(E) = (rk, c_1, ch_2+rk)$, where w_S denote the fundamental cocycle and $(r,l,s) \in H^*(S)$ denote degree 0, 2 and 4 components respectively. Note also that evenness of the intersection pairing for K3 surfaces implies that ch_2 and hence the Mukai vector is integral in this case.

Notation. Let $v \in H^*(S)$ be a cohomology class. Then We denote the moduli space of h-semistable sheaves with Mukai vector v by $M_h(v)$. Also let $M_h(v)^s \subset M_h(v)$ denote the open locus of stable sheaves. $M_h(v)^s \subset \mathrm{Spl}_S$ is open by Tudor's talk.

Recall. $M_h(\chi)$, the moduli space of h-semistable sheaves with reduced Hilbert polynomial χ is constructed using GIT on an open subset of a Quot scheme. Then, this implies it is quasi-projective. Hence, the following discussion on projectivity will essentially be properness.

Remark 2.1. Given $v = (r,l,s), v' = (r',l',s') \in H^*(S,\mathbb{Q})$ define the Mukai pairing $< v, v' > := ll' - rs' - s'r = -\int_S v^\vee v'$, where $v^\vee = (r,-l,s)$. Then a direct consequence of Grothendieck-Riemann-Roch theorem is

$$< v(E), v(E') > = -\chi(E,E')$$

for any $E, E' \in D^b(S)$. Applying this to the case $E' = \mathcal{O}(-n)$, where $\mathcal{O}(1)$ is the ample line bundle with first Chern class h, we obtain

$$\chi(E(n))/rk(E) = (h^2/2)n^2 + (l.h/r)n + (1 + s/r)$$

where $v(E) = (r,l,s)$. Hence, the Mukai vector determines the reduced Hilbert polynomial and (semi)stability can be checked directly from $\chi(E)/rk(E)$. This also implies $M_h(v) \subset M_h(\chi)$ open for some χ.

For more details on this see [Huy06] or [Muk87].

Corollary. *The dimension of $M_h(v)$ is equal to*

$$dimExt^1(E_0, E_0) = 2 - \chi(E_0, E_0) = 2+ < v, v >$$

when there exist an $E_0 \in M_h(v)^s$.

Mukai refers to some other paper to prove properness of $M_h(v)$. I think this can be easily proven using Harder-Narasimhan filtrations of families (properness was proven in this seminar by Tudor, see section 5 of his notes). Namely take a flat family of semistable sheaves over a formal punctured disc $B^o = Spec(K)$, where R is a DVR and K its field of fractions. Then, take a flat extension of this family to B, i.e. find a sheaf $\mathcal{E} \to B \times S$ restricting to previous family over the generic fiber. Find Harder-Narasimhan filtrations over the families, with R-flat semistable factors, and use some semicontinuity argument to conclude semistability at the generic fibers.(one may need to remove some torion from the special fibers etc., check the details yourself) See references in [Muk87].

Corollary (See [Muk87])**.** *$M_h(v)^s$ is projective if and only if every h-semistable sheaf is stable.*

Now, let us prove a numerical criteria that implies this:

Theorem 2.1. *Assume $v = (r, l, s) \in H^*(S)$ and $gcd(r, l.h, s) = 1$, then every semistable sheaf with Mukai vector v is stable, i.e. $M_h(v)^s$ is projective.*

Proof. Let E be semistable(hence torsion free) with $v(E) = v$ and assume that there exist $0 \subsetneq F \subsetneq E$. such that

$$\chi(E(n))/rk(E) = \chi(F(n))/rk(F)$$

for $n >> 0$. Note that the left hand side cannot be strictly smaller anyways, because of semistability. If we write $v(E) = (r, l, s)$ and $v(F) = (r', l', s')$, this equation implies that

$$(l.h)r' = (l'.h)r \text{ and } sr' = s'r$$

The assumption $gcd(r, l.h, s) = 1$ implies there exist $a, b, c \in \mathbb{Z}$ such that

$$ar + b(h.l) + cs = 1$$

But then multipliying this equation by r' and using previous equalities we get

$$r' = arr' + b(h.l)r' + cs'r = arr' + b(l'.h)r + cs'r$$

Hence, r divides r', so $rk(F) = rk(E)$. But then $F = E$, which is a contradiction, as the quotient E/F has to be semistable with the same phase. This finishes the proof. $\square$

Remark 2.2. Under the assumptions of the above theorem one can also show that the moduli space is fine and there exists a universal sheaf on it. See [Muk87].

Interesting applications of the above result follow from the assumption that $v^2 = 0$. Hence, we will assume this from now on. Note in this case, moduli space is a surface, and if we assume $gcd(r, h.l, s) = 1$ then it follows that we have a smooth projective surface $M_h(v)^s$. There are applications, where one starts with a given Mukai vector v and picks h generically so that this condition holds.

Theorem 2.2. *Let $v^2 = 0$ and suppose that $M_h(v)^s$ has a proper irreducible component. Then $M_h(v)^s$ is irreducible.*

Proof. We will restrict ourselves to the case when there exists a universal family $\mathcal{E} \to M_h(v)^s \times S$. But note this is not strictly necessary, Mukai works in a more general setting using the so called quasi-universal families, which always exist, the ideas are the same though. Also, let us assume that $M_h(v)^s$ is projective.

Now, let $M \subset M_h(v)^s$ be an irreducible component. By the smoothness, it is also a connected component. Assume $M \neq M_h(v)^s$. Let $\mathcal{E}$ be the universal sheaf over $M \times S$. We then have a transformation:

$$\Phi_{\mathcal{E}^\vee} : D^b(S) \to D^b(M)$$

such that $F \mapsto R\mathcal{H}om_M(\mathcal{E}, \pi_S^*(F))$. The idea is that this functor defines a morphism between the moduli space of sheaves on S, $M_h(v)$, and another moduli of sheaves, this time on M. Now, under this correspondence, we will show stable sheaves in M will go to point, whereas stable sheaves in $M_h(v) \setminus M$ will go to 0. But we will show Mukai vector v determines the Mukai vector of the image and it is different for skyscraper sheaves and 0.

Now let us take a closer look at what this does for the elements $E \in M_h(v)^s$. $R\mathcal{H}om_M(\mathcal{E}, \pi_S^*(E))$ is essentially the hom set between the universal family and the constant family that is isomorphic to E in the fibers of $M \times S \to M$.

Let us also note that if $E \neq E' \in M_h(v)^s$, then $RHom(E, E') = 0$. Hence, if $E \notin M$, it is orthogonal to any element of M and $R\mathcal{H}om_M(\mathcal{E}, \pi_S^*(E)) = 0$. On the other hand, if $E \in M$, then $R\mathcal{H}om_M(\mathcal{E}, \pi_S^*(E))$ is supported at the point $[E]$ of M corresponding to E. Instead of $R\mathcal{H}om$ we can take $R^i\mathcal{H}om$ and conclude the same thing. Mukai proves that it vanishes for $i = 0, 1$, again when we have $E \in M_h(v)^s$ (under the assumption that $\mathcal{E}$ is locally free this can be seen as $R\pi_{M*}$ of a sheaf supported at a single fiber of $M \times S \to M$. Then he uses a proposition proved there to conclude this result).

Now, it can be shown that the above transformation induces a map

$$\Phi^H_{\mathcal{E}^\vee} : H^*(S) \to H^*(M)$$

such that $v(F) \mapsto v(\Phi_{\mathcal{E}^\vee}(F))$ and this is just convolution with the the Mukai vector $v(\mathcal{E}^\vee)$ of $\mathcal{E}^\vee$ (we will elaborate more on this in the next section). Hence, it only depends on the Mukai vector. If $M \neq M_h(v)^s$, then v goes to 0 as $v = v(E)$ for some $E \notin M$. But as $M \neq \emptyset$, there exists $E' \in M$ and $v(E') \mapsto v(\mathcal{O}_m)$, for some $m \in M$. A simple application of Grothendieck-Riemann-Roch tells us that

$$td(S)ch(\mathcal{O}_S) = w_S$$

the fundamental cocyle, so $v(\mathcal{O}_m)$ is non-zero. This contradiction proves that $M = M_h(v)^s$, hence it is irreducible. $\square$

There is more to say about this moduli space, indeed it is a K3 surface under above assumptions but we will prove this in the next section in the framework of Fourier-Mukai transform.

3 Fourier-Mukai transform

In this section, we will introduce the Fourier-Mukai transform. The main reference that we are following is [Huy06]. Another good reference is [Orl97].

Recall the beautiful idea from the last section: To obtain transformations of the derived categories, we used an object $\mathcal{E}^\vee$ of $D^b(S \times M)$, giving rise to a transformation between some moduli of objects. This idea can be generalized to find non-trivial (exact, will always be exact) functors between derived categories. More precisely:

Definition 3.1. Let X and Y be smooth projective varieties and let P be an object of $D^b(X \times Y)$. Denote the projections from $X \times Y$ to X and Y by q and p respectively. Then define the **Fourier-Mukai transform**

$$\Phi_P : D^b(X) \to D^b(Y)$$

by $\Phi_P(\ ) = Rp_*(q^*(\ ) \overset{L}{\otimes} P)$. P is called the **kernel** of the twist.

Examples:

1. The functor $R\mathcal{H}om_M(\mathcal{E}, \pi_S^*(\ ))$ can be seen as a Fourier-Mukai transform with kernel $\mathcal{E}^\vee$, the derived dual.
2. Identity functor is a Fourier-Mukai tranform with kernel $\mathcal{O}_\Delta$. Similarly translation by 1, is of this type with kernel $\mathcal{O}_\Delta[1]$
3. More generally if $f : X \to Y$ is a morphism of varieties, then both Lf^* and Rf_* are of this type where the kernel is given by the structure sheaf of the graph of f(but the direction of the functor defined is changing, i.e. when taking Lf^* one must flip the roles of X and Y in the definition)
4. Let $\mathcal{L}$ be a line bundle. Then $(\ ) \otimes \mathcal{L}$ is a Fourier-Mukai transform(where the kernel is the push-forward of $\mathcal{L}$ to diagonal).
5. Composition of the Fourier-Mukai transforms are also Fourier-Mukai. Similarly, they always have Fourier-Mukai type right and left adjoint, whose kernel have explicit descriptions.

Note. This should convince you that most functors coming from geometry and algebra are of this type, although there are pathological counter-examples. In particular, every fully faithful exact functor is of this type, see [Orl97]. There are more general theorems about them being the functors lifting to dg enhancements, see [Kel06].

Remark 3.1. Fourier-Mukai transforms can be seen as the categorified version of the convolution in K-theory and cohomology. In particular they induce maps

$$\Phi_P^K : K(X) \to K(Y)$$

which is just the induced map between Grothendieck groups(hence convolution with $[P] \in D^b(X \times Y)$) and

$$\Phi_P^H : H^*(X) \xrightarrow{p_*(q^*(\ ).v(P))} H^*(Y)$$

the convolution with the Mukai vector of P. Note this map sends $v(F)$ to $v(\Phi_P(F))$ and the maps commute with $K(X) \xrightarrow{v(\ )} H^*(X)$ etc. These are both applications of Grothendieck-Riemann-Roch. For more details see [Huy06].

3.1 Derived categories of K3 surfaces

Both Fourier-Mukai transform and the moduli theory of the previous sections have applications to each other.

We first wish to show that under the assumptions of projectivity and $v^2 = 0$, $M_h(v)^s = M_h(v) = M$ is K3 surface. We already know it is a smooth, projective surface with an algebraic symplectic structure on it(hence its canonical bundle is trivial). Hence, it only remains to show that $H^1(M_h(v)) = 0$.

We already know that we have a map $\Phi_{\mathcal{E}}^H : H^*(M) \to H^*(S)$ (which is inverse to map $\Phi_{\mathcal{E}^\vee}^H$ of the previous section, but we skip this for the moment). This map is not preserving the degree, for instance it maps $v(\mathcal{O}_m) = (0, 0, 1)$ to v. But, as $v(\mathcal{E})$ is a sum of classes of type (p, p), we know that $\bigoplus_{p-q=i} H^{p,q}$ should be preserved under $\Phi_{\mathcal{E}}^H$, when we fix i. So if we can prove injectivity of $\Phi_{\mathcal{E}}^H$, then we would get $h^{1,0}(M) \leq h^{1,0}(S) = 0$ giving us what we want.

Now to prove injectivity, one can proceed in two ways:

1. One can show that $\Phi_{\mathcal{E}}$ is fully faithful. A cheap way of doing this using the theorem below

   Theorem 3.1 ([Orl97])**.** *A functor $\Phi_P : D^b(X) \to D^b(Y)$ as above is fully faithful if and only if*

   $$RHom^i(\Phi_P(\mathcal{O}_x), \Phi_P(\mathcal{O}_y)) = \begin{cases} 0 & x \neq y \\ 0 & i \notin [0, dim(X)] \\ \mathbb{C} & x = y, i = 0 \end{cases}$$

   A slightly weaker form of this theorem follows from the fact that the scyscraper sheaves generate the derived category as a triangulated category, hence, fully faithfullness can essentially be checked on them. To weaken the assumptions on the degree, requires more work, see [Huy06]. Once we know this theorem, we are essentially done: $\Phi_{\mathcal{E}}^H$ sends points to stable shaves and the assumptions of the theorem follow easily from the remarks in the proof of Theorem 2.2.

2. One can show $\Phi_{\mathcal{E}}^{H}$ is an isometry: There is an analogous Mukai pairing on M and it is a simple calculation to check that the map preserves the pairing. This also shows the result.

Ending remarks for the subsection:

1. One can indeed show that $\Phi_{\mathcal{E}}$ as above is an equivalence.

2. In general Φ_{P}^{H} is between the rational cohomologies. But when one has two K3 surfaces, it can be shown that $v(P)$ is integral, hence there is an induced map between integral cohomologies.

3. As mentioned before the degree or the Hodge structure is not preserved. But one can define a weight 2 Hodge structure on $H^*(S,\mathbb{Z}) =: \tilde{H}(S)$ such that $\tilde{H}(S)^{1,1} = H^0 \oplus H^{1,1} \oplus H^4$. Then if you have equivalences of derived categories of two K3 surfaces, the induced cohomological transform preserve this Hodge structure and it is a Hodge isometry.

4. One can then ask the inverse question: If $\tilde{H}(S_1)$ and $\tilde{H}(S_2)$ are Hodge isometric for two K3 surfaces S_1 and S_2, are their derived categories equivalent? The answer is yes. Indeed if we know that the isometry sends $(0,0,1)$ to $\pm(0,0,1)$ then it gives a Hodge isometry of $H^2(S_1,\mathbb{Z})$ and $H^2(S_2,\mathbb{Z})$, hence they are indeed isomorphic. If $(0,0,1)$ is sent to $v = (r,l,s)$ with $r \neq 0$, then one needs to fix it with another Hodge isometry interchanging $(0,0,1)$ with v that is induced by a derived equivalence. That is exactly given by the transformation between $D^b(M_h(v))$ and $D^b(S_2)$ as above. One still needs to choose h generically etc. For details again see [Huy06] or [Orl97].

5. This does not say anything about lifting an Hodge isometry though, we start with one, modify it on the way and then lift. One then wonders when one can lift and this question is deeper than it looks. A linear algebraic assumption on the effect of the isometry on positive four dimensional subspaces of $\tilde{H}$, whose quadratic form has signature $(4,20)$, can be shown to be sufficient for lifting, see [Huy06]. The converse is much harder, but it is shown, see [HMS09] for or [MS08].

3.2 Spherical Twist

We want to understand the autoequivalence group of derived categories and first thing to do is to produce non-trivial automorphisms of them, that are not of the type above. The main references for this section are [ST01] and [Huy06].

Intuition for such a thing can be taken from homological mirror symmetry conjecture, namely symplectomorphisms on the symplectic side should give rise to autoequivalences of the derived categories of coherent sheaves on the mirror. A simple symplectomorphism, that is not a priori in the identity component of the automorphism group is the Dehn twist and homological characterization of it gives rise to an example of a non-trivial equivalence in a more general setting, which works for derived categories of coherent sheaves as well. See [ST01].

Definition 3.2. Let X be a smooth projective variety and $P \in D^b(X)$. Then P is called **spherical** if:

1. $P \otimes \omega_X \cong P$
2. $Hom(P, P[i]) = \begin{cases} \mathbb{C} & i = 0,\, dim(X) \\ 0 & \text{otherwise} \end{cases}$

   i.e. $Hom^*(P, P)$ is isomorphic to cohomology of $S^{dim(X)}$ as a graded vector space.

Remark 3.2. Condition 1. always holds when X is K3 or abelian.

Examples:

1. Line bundles on K3 surfaces are spherical.
2. If X is polarized K3 and C is a (-2)-curve, then $\mathcal{O}_C(n)$ is spherical.
3. Skyscraper sheaves on smooth curves are spherical.
4. $\mathcal{O}_E$ for an elliptic curve E is spherical.

Now given this definition we can introduce spherical twist:

Definition 3.3 ([ST01],[Huy06])**.** Let X be smooth, projective and $P \in D^b(X)$ be spherical. Then define spherical twist functor $T_P : D^b(X) \to D^b(X)$ to be

$$T_P = Cone\big(RHom(P, (\ )) \overset{L}{\otimes} P) \to (\ )\big)$$

where the map is the natural evaluation map. If you decide to complain about lack of naturality of cones, you can take this to be the Fourier-Mukai transform with kernel

$$Cone((q^* P^\vee \otimes p^* P) \to \mathcal{O}_\Delta)$$

Theorem 3.2 ([ST01], [Huy06])**.** *If X is as above and P is spherical then T_P is an autoequivalence.*

Note. This actually works in a more general setting, as soon as one has some extra assumptions such as the non-degeneracy of certain pairings, which we have due to Serre duality. See [ST01].

Effect of T_P on cohomology As mentioned above, there is cohomological transform $T_P^H \curvearrowright H^*(X)$. But given that T_P is a cone of id and $RHom(P, (\ ))$, we get T_P^H to be the diffrence of the corresponding transforms. $id^H = id$ is pretty clear. The effect of $RHom(P, (\ ))$ on the cohomology, as can be predicted by looking its effect on the Mukai vectors, is multiplication by $v(P)^\vee$. Thus, one can see that

$$T_P^H : v \mapsto v + < v(P), v > v(P)$$

Note we are using generalized Mukai pairing $<, >$, here, which is just as before when X is K3. See [Huy06].

Example 3.1. If X is K3 and $C \subset X$ is a (-2)-curve, then $v(\mathcal{O}_C(-1)) = (0, [C], 0)$. Hence,

$$T_{\mathcal{O}_C(-1)}(v) = s_{[C]}(v) = v + < [C], v > [C]$$

is a reflection. Note this sort of reflection has its geometric importance as they can be used to move elements in the positive cone into ample cone. So, they can be used in problems related to lifting Hodge isometries. See [Huy06] and [MS08].

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