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P. Achar - Lect 3

Last time:

<u>Thm</u> (Mirković - Vilonen) $*$ on $\mathrm{Perv}_{G(\mathcal{O})}(Gr)$ is commutative

this is the main step in the proof of

<u>Thm</u> (Mirković - Vilonen) Let k be a comm. noeth. ring of finite global dim.

$$\begin{tikzcd} \mathrm{Perv}_{G(\mathcal{O})}(Gr, k) \arrow[rr] \arrow[dr, "H^*"'] & & \mathrm{Rep}(G^\vee_k) \arrow[dl, "\text{forget}"] \\ & \cancel{\mathrm{Vect}} \; k\text{-mod} & \end{tikzcd}$$

still over $\mathbb{C}$!!

split reduced gp. scheme/k whose root datum is dual to that of G

subtle pt: Need to show also that $\mathrm{Perv}_{G(\mathcal{O})}(Gr)$ has duals. Verdier duality does not quite do the trick, this rather corresponds to duality followed by Chevalley involution on $\mathrm{Rep}(G^\vee)$

Moral: Rep theory of $G^\vee_k$ is encoded in the topology of Gr_G.

<u>Today</u>: Applications to pos. char

I - Weyl modules

X_*^+ = dom. coweights for G
= dom. weights for $G^\vee$.

$\lambda \in X_*^+ \rightsquigarrow L_{\mathbb{C}}(\lambda)$ are irreps of $G^\vee_{\mathbb{C}}$

$L_k(\lambda)$ means $\underbrace{\text{irred. } G^\vee_k\text{-rep.}}_{\text{only makes sense when } k \text{ is a field}}$ of h. wt. λ

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unique up to iso of $G^\vee_{\mathbb{Z}}$-modules

Fact $\exists!$ minimal $G^\vee_{\mathbb{Z}}$-stable lattice

free $\mathbb{Z}$-module of max'l rank inside a $\mathbb{C}$-vector space

$$\Delta_{\mathbb{Z}}(\lambda) \subseteq L_{\mathbb{C}}(\lambda)$$

(minimal = ~~every $G^\vee_{\mathbb{Z}}$-stable lattice it contains is isomorphic to it)~~
~~ie minimal in the set of isoclasses of lattices~~
minimal containing a previously fixed highest weight vector!

this is a normalization condition

Defn: $\forall$ field k, the Weyl module of h. wt. λ is:

(Lusztig-Carter) 1974

$$\Delta_k(\lambda) := k \otimes_{\mathbb{Z}} \Delta_{\mathbb{Z}}(\lambda)$$

In general Weyl modules are not irreducible, but they have a unique simple quotient

$$\Delta_k(\lambda) \twoheadrightarrow L_k(\lambda)$$

Characters

$\mathrm{ch}\, L_{\mathbb{C}}(\lambda) =$ Weyl character formula (1925)

By construction

$\mathrm{ch}\, \Delta_k(\lambda) =$ Weyl character formula

Difficult question: $\mathrm{ch}\, L_k(\lambda) = ??$

II. MV cycles

Let $j_\lambda : \mathrm{Gr}_\lambda \hookrightarrow \mathrm{Gr}$ where, recall, Gr_λ is a $G(\mathcal{O})$-orbit

We have the simple perverse sheaves

$$IC_\lambda := IC(\mathrm{Gr}_\lambda, \Bbbk) = (j_\lambda)_{!*}(\underline{\Bbbk}[\dim \mathrm{Gr}_\lambda])$$

still makes sense for $\Bbbk = \mathbb{Z}$, but it's no longer simple

Recall that, by definition, $(j_\lambda)_{!*}(\underline{\Bbbk}[\dim \mathrm{Gr}_\lambda])$ is the image of

${}^pH$ = perverse cohomology

$$\underbrace{{}^pH^0((j_\lambda)_!(\underline{\Bbbk}[\dim \mathrm{Gr}_\lambda]))}_{I_!(\lambda,\Bbbk)} \longrightarrow \underbrace{{}^pH^0((j_\lambda)_*(\underline{\Bbbk}[\dim \mathrm{Gr}_\lambda]))}_{I_*(\lambda,\Bbbk)}$$

Define: $I_!(\lambda, \Bbbk)$ $\quad$ $I_*(\lambda,\Bbbk)$

<u>Thm</u> (Mirković-Vilonen) Under geom. Satake

1) For field coefficients

$$\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}, \Bbbk) \longrightarrow \mathrm{Rep}(G^\vee_\Bbbk)$$

$$IC(\mathrm{Gr}_\lambda, \Bbbk) \longmapsto L_\Bbbk(\lambda)$$
$$I_!(\lambda,\Bbbk) \longmapsto \Delta_\Bbbk(\lambda)$$
$$I_*(\lambda,\Bbbk) \longmapsto \nabla_\Bbbk(\lambda)$$
← dual Weyl module

2) For $\Bbbk = \mathbb{Z}$

$$I_!(\lambda, \mathbb{Z}) \overset{\simeq}{\twoheadrightarrow} IC(\mathrm{Gr}_\lambda, \mathbb{Z}) \hookrightarrow I_*(\lambda, \mathbb{Z})$$

This means: Can study Weyl modules, etc. using sheaf-theoretic properties of j_λ!

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T = max. torus of G

Recall $\quad Gr_T = X_*(T) \hookrightarrow Gr_G$
$$\lambda \longmapsto \underline{t}^{\lambda}$$

Take a Borel $B = TU$, U = max'l unip. grp

<u>Def-n</u> $\quad S_\lambda := U(\mathbb{K}) \cdot \underline{t}^{\lambda} \subseteq Gr_G \quad$ (Recall $\mathbb{K} = \mathbb{C}((t))$)

<u>Thm</u> (Mirković-Vilonen) $\lambda \in X_*^+$, $\mu \in X_*$

1) $S_\mu \cap Gr_\lambda \neq \emptyset \iff \mu$ occurs as a wt. of $L_{\mathbb{C}}(\lambda)$ or $\Delta_k(\lambda)$
If nonempty, then it is equidim. of dim $\langle \rho, \mu + \lambda \rangle$

2) The μ wt. space of

$Rep(G^\vee, k) \ni \mathbb{H}^\cdot(I_!(\lambda, k))$ is $H_c^{2\langle \rho, \mu+\lambda \rangle}(S_\mu \cap Gr_\lambda ; k)$

<u>Consequence</u>: The set of irred. components of $S_\mu \cap Gr_\lambda$ forms a basis for the μ wt. space of $\Delta_k(\lambda)$.

These components are called <u>MV-cycles</u>

<u>Note</u> 1) The set of MV-cycles is independent of k

2) This basis forms a crystal (Braverman-Gaitsgory)

3) Lots to say about combinatorics of MV cycles (Anderson, Kogan, Kamnitzer, Baumann...)

More on the exercises!

From now on, k = field of char. $p > h$, h = Coxeter # of G
(for GL_n, $h = n$)

III Principal block

Basic problem: compute $\mathrm{ch}\, L_k(\lambda)$

Related problem: compute $\mathrm{ch}\, T_k(\lambda)$, the indec. tilting module of ht wt λ.

Defn A rep. V of $G^\vee_k$ is tilting if both V & V^* have filtration by Weyl modules

Thm The indecomposable tilting modules are classified by highest weights

Examples:

For GL_n, take summands of $\otimes$'s of $\Lambda^k k^n$.

(In general, for large enough p, take summands of $\otimes$'s of fundamental repns)

Can reduce the basic problems to

$$\underset{\uparrow \atop \text{Principal block}}{\mathrm{Rep}_0}(G^\vee_k) := \left\langle L_k(\underbrace{w\rho - \rho + pw\lambda}_{(*)}) \;\middle|\; \begin{array}{l} w \in W \text{ (finite Weyl grp)} \\ \lambda \in X_* \\ w\rho - \rho + pw\lambda \in X_*^+ \end{array} \right\rangle$$

Lemma Dominant wts. of the permitted form $\longleftrightarrow X_*$

"λ" = Apply $(*)$ with w minimal element s.t. $w\lambda \in X_*^+$ $\longleftrightarrow \lambda$

Note $\mathrm{Rep}_0(G)$ is not a single block, in general

⑥ For SL_2

Dominant wts: $0, 1, 2, 3, \ldots$

Dominant wts. allowed in $\mathrm{Rep}_0(G)$: $\begin{array}{ccccccccc} 0, & p-2, & p, & 2p-2, & 2p, & 3p-2, & 3p, & 4p-2, & 4p, \ldots \\ \| & \| & \| & \| & \| & \| & \| & \| & \| \\ \text{"}0\text{"} & \text{"}-1\text{"} & \text{"}1\text{"} & \text{"}-2\text{"} & \text{"}2\text{"} & \text{"}-3\text{"} & \text{"}3\text{"} & \text{"}-4\text{"} & \text{"}4\text{"} \end{array}$

New problem: Compute $\mathrm{ch}\, L_k(\text{"}\lambda\text{"})$, $\mathrm{ch}\, T_k(\text{"}\lambda\text{"})$

IV Iwahori orbits

$$\begin{array}{ccc} \underline{G}(\mathbb{O}) & \xrightarrow[e]{t \mapsto 0} & \underline{G}(\mathbb{C}) \\ \cup\!\!\mid & & \cup\!\!\mid \\ e^{-1}(B) & \longrightarrow & B \leftarrow \text{Borel subgroup} \\ \| & & \\ I & \cdot\ \text{Iwahori subgroup} & \end{array}$$

e.g. in GL_2 $\qquad I = \begin{bmatrix} \mathbb{O}^\times & \mathbb{O} \\ t\mathbb{O} & \mathbb{O}^\times \end{bmatrix}$

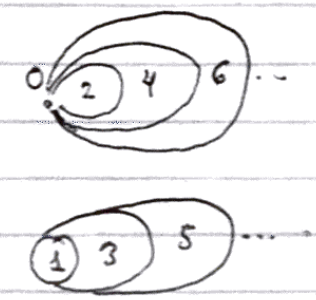
Thm 1) I-orbits on $Gr \longleftrightarrow X_*$

$\qquad I \cdot \underline{t}^\lambda \longleftrightarrow \lambda$

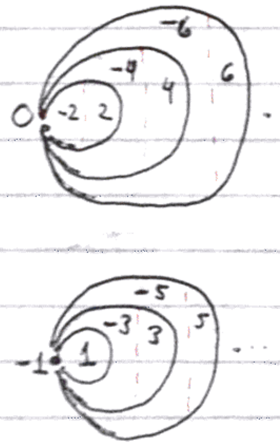
2) $\underline{G}(\mathbb{O}) \cdot \underline{t}^\lambda = Gr_\lambda = \bigsqcup_{\mu \in W\lambda} I \cdot \underline{t}^\mu$

⑦ Example $G = PGL_2$

Gr



$G(\mathcal{O})$-orbits



I-orbits

finite set, enough to determine all ch of irreducibles

Lusztig conjecture (1980) For "λ" restricted and $p > h$

Turns out to be false

$i + \dim$ of orbits

$$\mathrm{ch}\, L_k(``\lambda") = \sum (-1)^{\text{smth}} \underbrace{\dim IH^{i}_{I\cdot t^{\mu}}(I\cdot t^{\lambda})}_{\text{some kind of KL poly}} \underbrace{\mathrm{ch}\,\Delta_k(``\mu")}_{\text{Weyl char. formula}}$$

= stalk cohomology of $IC(I\cdot t^{\lambda})|_{x\in I\cdot t^{\mu}}$

- Proved in the 90s for $p \gg 0$ (unknown bound. Beautiful proof, uses conformal field theory)
- True for quantum groups in char. 0. Fiebig in '08 found an explicit (huge) bound e.g. for GL_9 $p > 10^{46}$
- Williamson '13: LOTS of counterexamples!!! for $h < p <$ Fiebig's bound

⑧ Actually, Williamson says that no polynomial in h is a bound (uses analytic number theory - growth of prime factors of Fibonacci numbers)

<u>Geometric plan to study characters</u>

$$\begin{tikzcd}[column sep=small]
 & & & \mathrm{Parity}_{IW}(\mathcal{Fl},\Bbbk) \arrow[dddlll, "\text{degrading}"', "\substack{\text{(conj. of}\\ \text{Riche-Williamson)}}"] \arrow[dr, phantom, "\subset"] & \textcolor{red}{G(\mathcal{K})/I} \arrow[d, phantom, "\textcolor{red}{\shortparallel}"] \\
 & & & & D^{mix}_{IW}(\mathcal{Fl},\Bbbk) \arrow[ddddl, leftrightarrow, "\substack{\wr\ \text{Koszul duality}\\ \text{(A., Makisumi}\\ \text{Riche Williamson)}}"] \\
\mathrm{Tilt}(\mathrm{Rep}_0(G^\vee)) \arrow[d, phantom, "\cap"] & & & \mathrm{Tilt}(\mathrm{Perv}^{mix}_{(I)}(\mathcal{Gr},\Bbbk)) \arrow[lll, "\text{degrading}"'] \arrow[d, phantom, "\cap"] & \\
\mathrm{Rep}_0(G^\vee) \arrow[d, phantom, "\cap"] & & & \mathrm{Perv}^{mix}_{(I)}(\mathcal{Gr},\Bbbk) \arrow[lll, "\text{degrading}\ \substack{\text{(conj. of}\\ \text{Finkelberg-Mirković)}}"'] \arrow[d, phantom, "\cap"] & \\
D^b\mathrm{Rep}_0(G^\vee) & D^b_{triv}(B^\vee) \arrow[l, "\sim", "\text{A-Riche}"', "\mathrm{Rind}_B^G"] & D^b\mathrm{Coh}^{G^\vee\times\mathbb{G}_m}(\tilde{\mathcal{N}}) \arrow[l, "\substack{\text{A-Riche}\\ \text{degrading}}"'] & D^{mix}_{(I)}(\mathcal{Gr},\Bbbk) \arrow[l, "\sim"', "\substack{\text{A. Riche}\\ \text{Mautner Riche}}"] & \\
 & & \mathrm{Rep}(G^\vee) \arrow[ull, "V^{(1)} \mapsfrom V"] \arrow[ul, bend right, "V^{(1)} \mapsfrom I_V"'] \arrow[u, hook, "\substack{V\otimes\mathcal{O}_{\tilde{\mathcal{N}}}\\ \mapsfrom\\ V}"'] & \mathrm{Perv}_{G(\mathcal{O})}(\mathcal{Gr},\Bbbk) \arrow[l, "\substack{\text{geom}\\ \text{Satake}}"] \arrow[u, hook] &
\end{tikzcd}$$

constructible for Iwahori orbits

$D^b_{triv}(B^\vee) \subseteq D^b\mathrm{Rep}(B^\vee)$, the subcat generated by $\Bbbk(p\lambda)$

<u>Degrading</u>: $F: \mathcal{C} \to \mathcal{B}$, triang cats & $\mathcal{C}$ has internal grading shift $\langle 1 \rangle$

F is <u>degrading</u> if

1) ~~[illegible]~~ The image generates $\mathcal{B}$ as triang. cat

2) $\bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, Y\langle n\rangle) \xrightarrow{\sim} \mathrm{Hom}(F(X), F(Y))$

Consequences of this setup.

NOTE

$$\operatorname{ch} L_{\Bbbk}(``\lambda") = \sum (-1)^{\text{smth}} IH^{i}_{I \cdot \underline{t}^{\mu}}(\overline{I \cdot \underline{t}^{\lambda}}, \Bbbk) \operatorname{ch} \Delta_{\Bbbk}(``\mu")$$

$$\operatorname{ch} T_{\Bbbk}(``\lambda") = \sum {}^{p}n_{\lambda,\mu}(1) \operatorname{ch} \Delta_{\Bbbk}(``\mu")$$

(This is now a thm. of Achar-Makisumi-Riche-Williamson)

antispherical p-KL-polynomial (hard to compute!)