Exercises on the affine Grassmannian

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Lecture 1

1. Let R be a commutative ring.

   (a) Show that $f \in R[[t]]$ is invertible if and only if its constant term is invertible in R.

   (b) Let $f \in R((t))$, say $f = a_k t^k + a_{k+1} t^{k+1} + \cdots$. Show that f is invertible if and only if there is an integer m such that $a_k, a_{k+1}, \ldots, a_m$ are nilpotent, and a_{m+1} is invertible. In particular, if R is a field, then $R((t))$ is a field.

   (c) Show that as an ind-variety, $\mathrm{Gr}_{\mathbb{G}_\mathrm{m}} \cong \mathbb{Z}$ (i.e., a discrete countable set). On the other hand, use the previous part to show that $\mathrm{Gr}_{\mathbb{G}_\mathrm{m}}$ is *not* a reduced ind-scheme (i.e., not a direct limit of reduced schemes).

2. Prove that every lattice in $\mathbf{K}^n$ is in the $\mathrm{GL}_n(\mathbf{O})$-orbit of a lattice with basis of the form

$$\{t^{a_1}\mathbf{e}_1, t^{a_2}\mathbf{e}_2, \ldots, t^{a_n}\mathbf{e}_n\}$$

   with $a_1 \geq a_2 \geq \cdots \geq a_n$. Moreover, the n-tuple $(a_1, a_2, \ldots, a_n)$ is uniquely determined. We therefore obtain a bijection

$$\{\mathrm{GL}_n(\mathbf{O})\text{-orbits on } \mathrm{Gr}_{\mathrm{GL}_n}\} \overset{\sim}{\longleftrightarrow} \{(a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n \mid a_1 \geq a_2 \geq \cdots a_n\}.$$

3. (a) Let $0 \leq k < n$, and consider the following dominant coweight for GL_n:

$$\varpi_k = (\underbrace{1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{n-k}).$$

   (These are called *minuscule* weights.) Show that Gr_{ϖ_k} is a closed $\mathrm{GL}_n(\mathbf{O})$-orbit in Gr, and that it is isomorphic to the (ordinary) Grassmannian of $(n-k)$-dimensional subspaces of $\mathbb{C}^n$.

   (b) Let $\lambda = (a_1, \ldots, a_n)$ be a dominant coweight for GL_n (so that $a_1 \geq \cdots \geq a_n$.) Let $m = a_1 - a_n$. Show that

$$\overline{\mathrm{Gr}_\lambda} = \left\{ \mathcal{L} \in \mathrm{Gr} \;\middle|\; \begin{array}{c} \text{there is a sequence of lattices } t^{a_1}\mathcal{L}^\circ = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_m = \mathcal{L} \\ \text{such that } t\mathcal{L}_i \subset \mathcal{L}_{i-1} \text{ and } \dim_\mathbb{C} \mathcal{L}_i/\mathcal{L}_{i-1} = j, \text{ where } a_j > a_1 - i \geq a_{j+1} \end{array} \right\}$$

   Moreover, Gr_λ is the open subset of $\overline{\mathrm{Gr}_\lambda}$ in which $\mathcal{L}_i = t^{-1}\mathcal{L}_{i-1} \cap \mathcal{L}$.

   If this is too difficult, start with this warm-up problem: Assuming that the description above is correct, show that every lattice in $\overline{\mathrm{Gr}_\lambda}$ has valuation $a_1 + \cdots + a_n$. Then do Problem 7a in the special case of minuscule coweights, then Problem 7c, then come back to this problem.

4. (Lusztig 1981) Consider the weight $\lambda = (n, 0, \ldots, 0)$ for GL_n. Let $\mathcal{M}$ be the open subset of $\overline{\mathrm{Gr}_\lambda}$ consisting of lattices $\mathcal{L}$ such that $\mathcal{L} \cap (t^{-1}\mathbb{C}[t^{-1}])^n = 0$. (In other words, $\mathcal{L}$ contains no vector whose coordinates involve only strictly negative powers of t.) Show that $\mathcal{M}$ is isomorphic to the affine variety $\mathcal{N}$ of $n \times n$ nilpotent matrices.

   (*Hint:* Let $\mathcal{L} \in \mathcal{M}$ and let $v \in \mathcal{L}$. Write v as $\sum_{j > -n} v_j t^j$, where $v_j \in \mathbb{C}^n$. The assumption that $\mathcal{L} \cap (t^{-1}\mathbb{C}[t^{-1}])^n = 0$ implies that $v_{-n+1}, v_{-n+2}, \ldots, v_{-1}$ are determined by v_0. In fact, there is a linear map $x : \mathbb{C}^n \to \mathbb{C}^n$ such that $v_{-k} = x^k v_0$, and $x^n = 0$. The assignment $\mathcal{L} \mapsto x$ gives the desired map $\mathcal{M} \to \mathcal{N}$.)

5. Two lattices $\mathcal{L}$ and $\mathcal{L}'$ in $\mathbf{K}^n$ are said to be *homothetic* if there is a nonzero scalar $s \in \mathbf{K}^\times$ such that $\mathcal{L} = s\mathcal{L}'$. Show that $\mathrm{Gr}_{\mathrm{PGL}_n}$ can be identified with the set of homothety classes of lattices.

6. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{f}_1, \ldots, \mathbf{f}_n\}$ be the standard basis for $\mathbf{K}^{2n}$. Equip $\mathbf{K}^{2n}$ with the bilinear form:

$$\langle \mathbf{e}_i, \mathbf{e}_j\rangle = \langle \mathbf{f}_i, \mathbf{f}_j\rangle = 0 \text{ for all } i,\ j, \qquad \langle \mathbf{e}_i, \mathbf{f}_j\rangle = -\langle \mathbf{f}_j, \mathbf{e}_i\rangle = \delta_{ij}.$$

   (a) A *symplectic lattice* is a lattice $\mathcal{L} \subset \mathbf{K}^{2n}$ such that $\langle\cdot,\cdot\rangle$ restricts to a perfect $\mathbf{O}$-valued pairing on $\mathcal{L}$. Show that $\mathrm{Gr}_{\mathrm{Sp}_{2n}}$ can be identified with the set of symplectic lattices.

   (b) Give a lattice-theoretic description of $\mathrm{Gr}_{\mathrm{PSp}_{2n}}$. (This affine Grassmannian has two connected components, one of which can be identified with $\mathrm{Gr}_{\mathrm{Sp}_{2n}}$. What does the other component consist of?)

   (c) Recall that $\mathrm{Sp}_2 = \mathrm{SL}_2$. How is this related to the description of $\mathrm{Gr}_{\mathrm{SL}_2}$ from the lecture?

   (d) Give analogous descriptions of the affine Grassmannians of SO_{2n+1} and SO_{2n}.

Lecture 2

7. The following questions deal with the convolution space for GL_n. It might be a good idea to start with the special case where the coweights are minuscule.

   (a) Let $\lambda = (a_1, \ldots, a_n)$ and $\mu = (b_1, \ldots, b_n)$ be two dominant coweights. Let $m = b_1 - b_n$. Show that $\overline{\mathrm{Gr}_\lambda} \tilde{\times} \overline{\mathrm{Gr}_\mu} \subset \mathrm{Gr} \tilde{\times} \mathrm{Gr}$ can be identified with the set

$$\left\{ (\mathcal{L}, \mathcal{L}') \;\middle|\; \begin{array}{c} \mathcal{L} \in \overline{\mathrm{Gr}_\lambda}, \text{ and} \\ \text{there is a sequence of lattices } t^{b_1}\mathcal{L} = \mathcal{L}'_0 \subset \mathcal{L}'_1 \subset \cdots \subset \mathcal{L}'_m = \mathcal{L}' \\ \text{such that } t\mathcal{L}'_i \subset \mathcal{L}'_{i-1} \text{ and } \dim_{\mathbb{C}} \mathcal{L}'_i/\mathcal{L}'_{i-1} = j, \text{ where } b_j > b_1 - i \geq b_{j+1}. \end{array} \right\}$$

   Moreover, show that the image of $m : \overline{\mathrm{Gr}_\lambda} \tilde{\times} \overline{\mathrm{Gr}_\mu} \to \mathrm{Gr}$ is $\overline{\mathrm{Gr}_{\lambda+\mu}}$.

   (b) Let $\lambda^{(1)}, \ldots, \lambda^{(k)}$ be a sequence of dominant coweights. Generalize the previous part to give a description of $\overline{\mathrm{Gr}_{\lambda^{(1)}}} \tilde{\times} \cdots \tilde{\times} \overline{\mathrm{Gr}_{\lambda^{(k)}}}$.

   In fact, upon further reflection, I think you should start with the following problem:

   (c) Let $\lambda = (a_1, \ldots, a_n)$, and let $m = a_1 - a_n$. Define integers $k_{m-1}, k_{m-2}, \ldots, k_1, k_0$ by the condition that $a_{k_i} \geq a_1 - i > a_{k_i+1}$. Show that

$$m : \mathrm{Gr}_{(a_n,\ldots,a_n)} \tilde{\times} \mathrm{Gr}_{\varpi_{k_{m-1}}} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_{\varpi_{k_0}} \to \overline{\mathrm{Gr}_\lambda}$$

   is a resolution of singularities.

8. Determine the fibers of the following convolution morphisms for GL_2:

   (a) $m : \mathrm{Gr}_{(1,0)} \tilde{\times} \mathrm{Gr}_{(1,0)} \to \overline{\mathrm{Gr}_{(2,0)}}$. (*Answer:* For $x \in \mathrm{Gr}_{(2,0)}$, the fiber is a point. For $x \in \mathrm{Gr}_{(1,1)}$, the fiber is isomorphic to $\mathbb{P}^1$.)

   (b) $m : \mathrm{Gr}_{(1,0)} \tilde{\times} \mathrm{Gr}_{(1,0)} \tilde{\times} \mathrm{Gr}_{(1,0)} \to \overline{\mathrm{Gr}_{(3,0)}}$. (*Answer:* For $x \in \mathrm{Gr}_{(3,0)}$, the fiber is a point. For $x \in \mathrm{Gr}_{(2,1)}$, the fiber looks like two copies of $\mathbb{P}^1$ meeting at a point.)

   (c) Carry out the same computation for some other weights of your own choosing. If you are feeling adventurous, go up to GL_3.

9. Let Φ^+ be the set of positive roots, and let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$. The *q-analogue of the Kostant partition function* is the family of polynomials $P_\nu(q)$ (where $\nu \in \mathbf{X}_*$ and q is an indeterminate) given by the generating function

$$\prod_{\alpha\in\Phi^+} \frac{1}{1 - qe^\alpha} = \sum_{\nu\in\mathbf{X}_*} P_\nu(q)e^\nu.$$

For $\lambda \in \mathbf{X}_*^+$ and $\mu \in \mathbf{X}_*$, the *q-analogue of the weight multiplicity* is the polynomial $M_\lambda^\mu(q)$ given by

$$M_\lambda^\mu(q) = \sum_{w \in W} (-1)^{\ell(w)} P_{w(\lambda+\rho)-(\mu+\rho)}(q).$$

Recall from the lecture that Lusztig proved that when λ and μ are both dominant, we have

$$M_\lambda^\mu(q) = \sum_{i \geq 0} \operatorname{rank} \mathcal{H}^{-\dim \mathrm{Gr}_\mu - i}(\mathrm{IC}_\lambda|_{\mathrm{Gr}_\mu}) q^{i/2}.$$

Compute $P_\nu(q)$ and $M_\lambda^\mu(q)$ in general for SL_2. Check that $M_\lambda^\mu(1)$ is always the dimension of the μ-weight space of $L(\lambda)$ (even if μ is not dominant!). Check that when μ is dominant, $M_\lambda^\mu(q)$ has nonnegative coefficients.

Here are the answers: identifying $\mathbf{X}_*$ with $\mathbb{Z}$, we have

$$P_\nu(q) = \begin{cases} q^{\nu/2} & \text{if } \nu \in 2\mathbb{Z}_{\geq 0}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$M_\lambda^\mu(q) = \begin{cases} 0 & \text{if } \mu > \lambda \text{ or } \lambda \not\equiv \mu \pmod 2, \\ q^{(\lambda-\mu)/2} & \text{if } -\lambda \leq \mu \leq \lambda \text{ and } \lambda \equiv \mu \pmod 2, \\ q^{(\lambda-\mu)/2} - q^{(-\lambda-\mu-2)/2} & \text{if } \mu \leq -\lambda - 2 \text{ and } \lambda \equiv \mu \pmod 2. \end{cases}$$

10. (This question requires some familiarity with calculating with perverse sheaves.) Use Problem 8 to compute $\mathrm{IC}_{(1,0)} \star \mathrm{IC}_{(1,0)}$ and $\mathrm{IC}_{(1,0)} \star \mathrm{IC}_{(1,0)} \star \mathrm{IC}_{(1,0)}$. Use these calculations to determine the stalks of $\mathrm{IC}_{(2,0)}$ and $\mathrm{IC}_{(3,0)}$. Check that these agree with the q-analogue of the weight multiplicity that you computed in the previous question.

11. In the affine Grassmannian of GL_3, determine the space $S_{(0,0,0)} \cap \overline{\mathrm{Gr}_{(1,0,-1)}}$. This variety should have two irreducible components, each of dimension 2. The two components provide a basis for the zero weight space of the adjoint representation of GL_3.

    (*Hint:* One could equivalently work in $S_{(1,1,1)} \cap \overline{\mathrm{Gr}_{(2,1,0)}}$. For the latter, it might be helpful to start by looking at the open subset $\mathcal{M} \subset \overline{\mathrm{Gr}_{(3,0,0)}}$ from Problem 4. Then this MV cycle calculation turns into a problem about 3×3 nilpotent matrices.)

Lecture 3

12. Let $\check{B}$ be the Borel subgroup of $\check{G}$ corresponding to the *negative* roots, and let $\check{\mathfrak{u}}$ be the Lie algebra of its unipotent radical. It follows from results of Brylinski that for $\lambda \in \mathbf{X}_*^+$ and $\mu \in \mathbf{X}_*$, we have

$$M_\lambda^\mu(q) = \sum_{n \geq 0} \left(\sum_{i \geq 0} (-1)^i \dim \operatorname{Ext}^i_{\check{B}}(L(\lambda), \operatorname{Sym}^n(\check{\mathfrak{u}}^*) \otimes \mathbb{C}_\mu) \right) q^n.$$

    Prove this directly for $\check{G} = \mathrm{SL}_2$. (*Hint:* For this group, nonzero Ext-groups can occur only for $i = 0, 1$. If μ is dominant, then only $i = 0$ can occur.)

13. The first part of this question requires some familiarity with perverse sheaves. However, you can treat the first part as a "black box" and then work out the second part.

    (a) Let $G = \mathrm{PGL}_2$, and let $\lambda, \mu \in \mathbf{X}_*$. Assume that $I \cdot \mathbf{t}_\mu \subset \overline{I \cdot \mathbf{t}_\lambda}$. Prove that $\mathrm{IC}(\overline{I \cdot \mathbf{t}_\lambda})|_{I \cdot \mu}$ is isomorphic to the shifted constant sheaf $\underline{\mathbb{C}}[\dim I \cdot \mathbf{t}_\lambda]$. (*Hint:* First treat the case where λ is dominant, using the the calculation of $M_\lambda^\mu(q)$ from Problem 9. Then, if λ is not dominant, show that $\overline{I \cdot \mathbf{t}_\lambda}$ is isomorphic to the closure of a dominant I-orbit on the other connected component of $\mathrm{Gr}_{\mathrm{PGL}_2}$.)

    (b) Use the result of the previous part to compute the characters of simple modules in the principal block of the quantum group $U_q(\mathfrak{sl}_2)$ specialized at a root of unity.