DAY 3 EXERCISES

1. Modules over quantizations

**Exercise 1.1** (Lemma 1.1 in the notes). Let \( M = \bigcup_{i \in \mathbb{Z}} M_{\leq i} = \bigcup_{i \in \mathbb{Z}} M_{\geq i} \) be two good filtrations. Then, there exist integers \( a, b \) such that \( M_{\leq i + a} \subset M_{\leq i} \subset M_{\geq i + b} \) for all \( i \in \mathbb{Z} \).

Hint: First, show that \( M \) is finitely generated. Then show that if \( M_{\leq i} \) is a good filtration on \( m \), then there exists \( m_1, \ldots, m_n \in M \) and integers \( d_1, \ldots, d_n \) such that \( M_{\leq i} = \mathcal{A}_{\leq i - d_1} m_1 + \cdots + \mathcal{A}_{\leq i - d_n} m_n \). The result now easily follows.

**Exercise 1.2.** Let \( M, N \) be finitely generated modules over a filtered algebra \( \mathcal{A} \). Assume that good filtrations are given on \( M \) and \( N \). Show that \( \text{Ext}^i_{\mathcal{A}}(M, N) \) admits a filtration for all \( i \) and, moreover, that we have a natural inclusion \( \text{gr} \text{Ext}^i_{\mathcal{A}}(M, N) \hookrightarrow \text{Ext}^i_{\mathcal{A}}(\text{gr} M, \text{gr} N) \).

Hint: Work at the level of Rees algebras and modules.

2. Localization theorems

**Exercise 2.1.** Finish the proof of Lemma 2.1 in the notes. That is, show that if \( \text{Loc}^q_\lambda \) is essentially surjective and \( \Gamma \) is exact, then abelian localization holds for \((\theta, \lambda)\).

Hint: To show that \( M \to \Gamma(\text{Loc}^q_\lambda(M)) \) is an isomorphism, use the fact that we know that this is an isomorphism when \( M \) is a free \( \mathcal{A}_\lambda \)-module. Then you can show that \( \mathcal{A}_\lambda^q \otimes_{\mathcal{A}_\lambda} \Gamma(M) \to M \) is an isomorphism (recall that the condition in the exercise is equivalent to saying that every \( M \in \text{Coh}(\mathcal{A}_\lambda^q) \) is generated by its global sections and has no higher cohomology).