

Commuting nilpotent matrices and generic Jordan type

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University of Southern California, April 7, 2014

Abstract

The similarity class of an $n \times n$ nilpotent matrix B over a field k is given by its *Jordan type*, the partition P_B of n , given by the sizes of the Jordan blocks. The variety \mathcal{N}_B parametrizing nilpotent matrices that commute with B is irreducible, so there is a partition $Q = \Omega(P)$ that is a generic Jordan type for A in \mathcal{N}_B .

P. Oblak, and T. Košir showed that $\Omega(P)$ is a “Rogers-Ramanujan” (RR) partition, whose parts differ pairwise by at least two. A recursive conjecture of P. Oblak for $\Omega(P)$ appears to be shown by R. Basili, after progress by P. Oblak, L. Khatami and others. Nevertheless, given Q , finding the set $\Omega^{-1}(Q)$ is open in general.

We show a “Table Conjecture” concerning the set of P such that $\Omega(P) = Q$ when Q has two parts. We extend this to state a “Box Conjecture” for $\Omega^{-1}(Q)$ for all RR partitions Q .

In this talk we present these, and ask “why?”

Section 1: The map $\Omega : P \rightarrow \Omega(P)$

Definition (Nilpotent commutator \mathcal{N}_B)

$V \cong k^n$ vector space over an infinite field k .

$A, B \in \text{Mat}_n(k) = \text{Hom}_k(V, V)$;

$P \vdash n$ partition of n ;

$J_P =$ Jordan block matrix of Jordan type P

$\mathcal{C}_B \subset \text{Mat}_n(k)$ centralizer of B .

$\mathcal{N}_B \subset \mathcal{C}_B$: the variety of nilpotent elements of \mathcal{C}_B .

$P_A =$ Jordan type of A .

Fact: \mathcal{N}_B is an irreducible variety [Bas1, BI].

Def: $\Omega(P) = P_A$ for A generic in \mathcal{N}_B , $B = J_P$.

Problem 1. Given the partition P , determine $\Omega(P)$

Fact. $\Omega(P)$ is Rogers-Ramanujan (RR): the parts of $\Omega(P)$ differ by at least two.

Problem 2. Given the RR partition Q determine $\Omega^{-1}(Q)$.

Prob. 1: Recursive conjecture of P. Oblak (2008) for $\Omega(P)$: work of P. Oblak, P. Oblak-T.Košir, L. Khatami, I-Khatami, R. Basili.

Prob 2: Table conjecture of P. Oblak and R. Zhao (2012,2013) is shown for $Q = (u, u - r), r \geq 2$. Box conjecture for $\Omega^{-1}(Q)$ is open for Q RR with $k > 2$ parts..

Classical problem: but not studied classically. Connected with Hilbert scheme work of J. Briançon, M. Granger, R. Basili, V. Baranovsky, A. Premet. See N. Ngo-K. Šivic.

In 2006, three groups began to work on the $P \rightarrow \Omega(P)$ problem, independently

- P. Oblak and T. Košir (Ljubljana)

- D. Panyushev (Moscow)

- R. Basili, I.-, and L.Khatami (Perugia, Boston).

Links to work of E. Friedlander, J. Pevtsova, A. Suslin, on representations of Abelian p -groups [FrPS,CFrP].

Definition (Almost rectangular)

Let $B = J_{(n)}$, and denote by $[n]^k = P_{B^k}$.

For $n = kq$, $[n]^k = (q^k) = (q, q, \dots, q)$.

For $n = kq + r$, $0 < r < k$, $[n]^k = ((\lceil n/k \rceil)^r, (\lfloor n/k \rfloor)^{k-r})$

Here $[n]^k$ has k parts that differ at most by 1.

We term $[n]^k$ *almost rectangular (AR)*.

Ex. $n = 5$,

$[5]^2 = (3, 2)$, $[5]^3 = (2, 2, 1)$, $[5]^4 = (2, 1, 1, 1)$, $[5]^5 = (1, 1, 1, 1, 1)$.

Theorem ((R. Basili) Ω for $r_P = 1$)

For $P = [n]^k$, $\Omega(P) = [n]$ and $\Omega^{-1}([n]) = \{[n]^k, 1 \leq k \leq n\}$

Example ($P = (3, 1)$ does not commute with $[4]$.)

$$\begin{array}{ccccc}
 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 \\
 & & A & &
 \end{array}
 \qquad
 \begin{array}{ccccc}
 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 \\
 & & A^2 & &
 \end{array}
 \qquad
 \begin{array}{ccccc}
 0 & 1 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 0 \\
 & & J_P & &
 \end{array}$$

Figure : $A = J_{[5]}$, A^2 , and J_P where $P = [5]^2 = (3, 2)$.
 Here A^2 is conjugate to J_P .

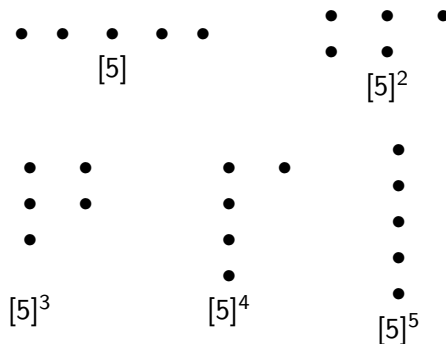


Figure : The AR partitions of 5.

Theorem (R. Basili [Bas1])

$\Omega(P)$ has r_P parts, where $r_P =$ minimum number of AR partitions P_i such that $P = \bigcup P_i$.

Theorem (R. Basili and I.- [BI])

$\Omega(P) = P \Leftrightarrow P$ is RR: the parts of P differ pairwise by at least 2.

Def. We call a $P \mid \Omega(P) = P$ “stable”

also “super-distinct” or “Rogers-Ramanujan” [AlBe, An].

Example

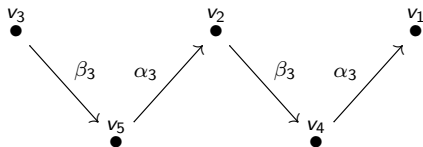
$$P = (3, 1), \quad \Omega(P) = (3, 1).$$

$$P = (\underbrace{5, 4}, \underbrace{3, 3}, 2, 1), \quad \Omega(P) = (12, 5, 1).$$

Poset \mathcal{D}_P

Rows of vertices: Span the maximal irreducible B -invariant subspaces of V : each row corresponds to a part of P .

Arrows: non-zero elements in $A \in \mathcal{U}_B$ (max subalgebra of \mathcal{N}_B).



$$A = \left(\begin{array}{ccc|cc} 0 & x_{\alpha_3\beta_3} & x_{(\alpha_3\beta_3)^2} & x_{\alpha_3} & x_{\alpha_3\beta_3\alpha_3} \\ 0 & 0 & x_{\alpha_3\beta_3} & 0 & x_{\alpha_3} \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & x_{\beta_3} & x_{\beta_3\alpha_3\beta_3} & 0 & x_{\beta_3\alpha_3} \\ 0 & 0 & x_{\beta_3} & 0 & 0 \end{array} \right), v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix}$$

Figure : Generic element A of \mathcal{U}_B , $B = J_P$ where $P = (3, 2)$.

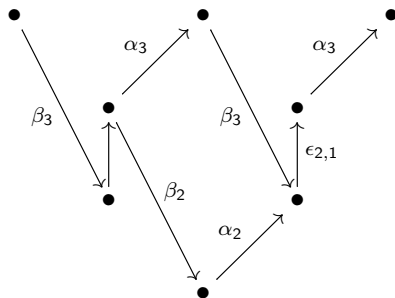


Figure : $\text{Diag}(\mathcal{D}_P)$ for $P=(3,2,2,1)$.

$$\left(\begin{array}{ccc|cc|cc|c} 0 & x_{c_3} & x_{(c_3)^2} & x_{\alpha_3} & x_{\alpha_3 c_2} & x_{\alpha_3 e_{21}} & x_{\alpha_3 c'_2} & x_{\alpha_3 e_{21} \alpha_2} \\ 0 & 0 & x_{c_3} & 0 & x_{\alpha_3} & 0 & x_{\alpha_3 e_{21}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & x_{e_{21} \beta_3} & x_{43} & 0 & x_{c_2} & x_{e_{21}} & x_{c'_2} & x_{\alpha_3 e_{21}} \\ 0 & 0 & x_{e_{21} \beta_3} & 0 & 0 & 0 & x_{e_{21}} & 0 \\ \hline 0 & 0 & x_{63} & 0 & x_{\alpha_2 \beta_2} & 0 & x_{\alpha_2 \beta_2 e_{21}} & x_{\alpha_2 \beta_2} \\ 0 & 0 & x_{\beta_3} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & x_{\beta_2 e_{21} \beta_3} & 0 & x_{\beta_2} & 0 & x_{\beta_2 e_{21}} & 0 \end{array} \right)$$

$$x_{63} = x_{\alpha_2 \beta_2 e_{21} \alpha_3}$$

Figure : Generic element A of \mathcal{U}_B for $P = (3, 2, 2, 1)$.

Relation with Artin algebras

Let $PCN_n = \{\text{pairs } A, B \text{ of } n \times n \text{ nilp. matrices, } [A, B] = 0\}$.

V. Baranovsky (2001) showed that PCN_n is irreducible.

When $\text{char } k = 0$ he used a result of J. Briançon (1978) and a proof of M. Granger (1983) that the Hilbert scheme $\text{Hilb}^n k\{x, y\}$ parametrizing length- n Artin algebras is irreducible.

R. Basili (2003, $\text{char } k \geq n/2$) and A. Premet (2003, all infinite k) showed the irreducibility of PCN_n directly.

This implies the irreducibility of $\text{Hilb}^n k\{x, y\}$ for all infinite k .

Pencil Lemma (I.-R. Basili)

Let $\mathcal{A} = k[A, B]$ be an Artin algebra with FHS $H = H(\mathcal{A})$ and $\text{char } k \geq n = \dim_k \mathcal{A}$. Then $P_C = H^\vee$, the conjugate of H , for $C = A + \lambda B$, $\lambda \in k$ generic.

Theorem (P. Oblak and T.Košir [KO])

For $A \in \mathcal{N}_B$ generic, the Artin algebra $k[A, B]$ is *Gorenstein*, so a complete intersection (CI).

Proof. Uses an involution of the poset \mathcal{D}_P of \mathcal{N}_B . See also [BIK, Thm. 2.20].

Corollary (ibid. with F.H.S. Macaulay [Mac])

$\Omega(P)$ is stable! ($\Omega(P)$ is RR: Parts differ pairwise by at least two)

Proof. After Macaulay, if \mathcal{A} is CI, the jumps $e_i = H_i - H_{i+1}$ of $H = H(\mathcal{A})$ are each less or equal 1, which implies H^\vee is RR.

Example

For $H = (1, 2, 3, 4, 3, 2, 2, 1)$, $H^\vee = (8, 6, 3, 1)$, which is RR.

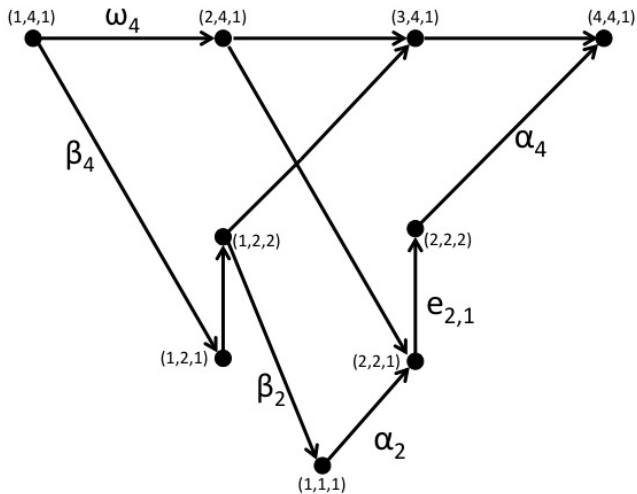


Diagram of the poset \mathcal{D}_P and maps, $P = (4, 2, 2, 1)$.

Definition (Poset \mathcal{D}_P [Obl1, KO, BIK, Kh1])

Let $P \vdash n$, $P = (\dots i^{n_i} \dots)$, $S_P = \{i \mid n_i > 0\}$. The poset \mathcal{D}_P has rows of the Ferrers graph of P , each row centered on the y -axis.

There are n_i rows of length i :

$$(u, i, k), 1 \leq u \leq i, 1 \leq k \leq n_i.$$

Let i^-, i^+ be the next smaller, next larger elements of S_P . The edges of \mathcal{D}_P correspond to *elementary maps*:

Maps and edges of the diagram \mathcal{D}_P

- (i) $\beta_i = \beta_{i,i^-} : (u, i, n_i) \rightarrow (u, i^-, 1)$ for $u \leq i^-$.
- (ii) $\alpha_i = \alpha_{i^-,i} : (u, i^-, n_{i^-}) \rightarrow (u + i - i^-, i, 1)$.
- (iii) $e_{i,k} : (u, i, k) \rightarrow (u, k, k + 1), 1 \leq u_i \leq i, 1 \leq k < n_i$.
- (iv) When i is isolated: $i - 1 \notin S_P, i + 1 \notin S_P$,
 $\omega_i : (u, i, n_i) \rightarrow (u + 1, i, n_i)$ for $1 \leq u < i$.

(Each map is 0 on the points of \mathcal{D}_P not listed)

The *diagram* of a poset has the covering edges only.

The \mathcal{D}_P is related to a maximum nilpotent subalgebra $\mathcal{U}_B \subset N_B, B = J_P$: $v < v'$ if $\exists A \in \mathcal{U}_B \mid A_{v,v'} \neq 0$.

Def: U -chain in \mathcal{D}_P determined by an AR $P' \subset P$: a chain that includes all vertices of \mathcal{D}_P from an AR subpartition P' , + two tails.

The first tail descends from the source of \mathcal{D}_P to the AR chain of P' , and the second tail ascends from the AR chain to the sink of \mathcal{D}_P .

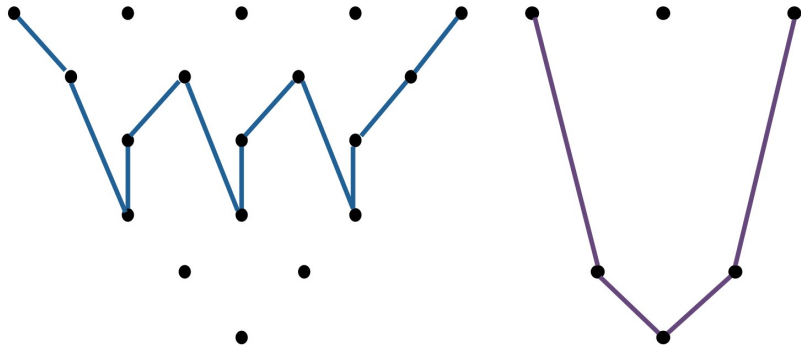


Figure : U -chain C_4 : $P = (5, 4, 3, 3, 2, 1)$ and new U -chain of $P' = (3, 2, 1)$. [Source: LK NU GASC talk 2013]

Oblak Recursive Conjecture

One obtains $\mathfrak{Q}(P)$ from \mathcal{D}_P :

- (i) Let C be a longest U -chain of \mathcal{D}_P . Then $|C| = q_1$, the biggest part of $\mathfrak{Q}(P)$.

- (ii) Remove the vertices of C from \mathcal{D}_P , giving a partition $P' = P - C$. If $P' \neq \emptyset$ then $\mathfrak{Q}(P) = (q_1, \mathfrak{Q}(P'))$ (Go to (i).).

Warning! The poset $\mathcal{D}_{P'}$ is not a subposet of \mathcal{D}_P .

Theorem (P. Oblak [Obl1] – Index of $\Omega(P)$)

The index of $\Omega(P)$ is the length of the longest U-chain C of \mathcal{D}_P .

Theorem (L. Khatami [Kh1] – $Ob(P) = \lambda_U(\mathcal{D}_P)$)

The partition $Ob(P)$ obtained by Oblak recursion is independent of choices of AR subpartitions, and $Ob(P) = \lambda_U(\mathcal{D}_P)$, obtained in the same way as $\lambda(\mathcal{D}_P)$ but using U-chains.¹

Work of I-L. Khatami (1/2 Oblak Rec Conj), L. Khatami (smallest part of $\Omega(P)$), and R. Basili (Oblak Rec Conj for char $k = 0$, 2014) shows the Recursive Conjecture.

¹A theory of E.R. Gansner, D. Kleitman, C. Greene, S. Poljak, T. Britz and S. Fomin assigns a partition $\lambda(\mathcal{P})$, using the lengths of multichains of a poset \mathcal{P}

Section 2: Table conjecture for $\mathfrak{Q}^{-1}(Q)$.

The set $\mathfrak{Q}^{-1}(Q)$ is mysterious, even for $Q = (u, u - r)$, $r \geq 2$ where $P \rightarrow \mathfrak{Q}(P)$ is explicit. P. Oblak (2012) [Obl2] and R. Zhao (2013) made a very beautiful conjecture.

Table conjecture for $\mathfrak{Q}^{-1}(Q)$ (P. Oblak, R. Zhao)

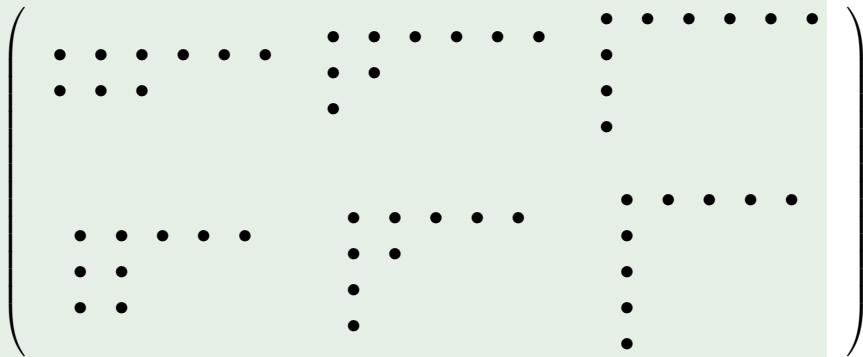
The elements of $\mathfrak{Q}^{-1}(Q)$, $Q = (u, u - r)$, $r \geq 2$ form a $(r - 1) \times (u - r)$ table $\mathcal{T}(Q)$ such that $\mathcal{T}(Q)_{i,j}$ has $i + j$ parts.

[P. Oblak: $\# \mathfrak{Q}^{-1}(Q) = (r - 1)(u - r)$; R. Zhao: table $\mathcal{T}(Q)$].

Example (Table $\mathcal{T}(Q)$ for $Q = (6, 3)$)

Let $Q = (6, 3)$.

$$\mathcal{T}(Q) = \begin{pmatrix} (6, 3) & (6, [3]^2) & (6, [3]^3) \\ (5, [4]^2) & (5, [4]^3) & (5, [4]^4) \end{pmatrix} = \begin{pmatrix} A & A & A \\ B & B & B \end{pmatrix}$$



Definition (Type A,B,C partitions in $\Omega^{-1}(Q)$)

Let $Q = (u, u - r)$, $r \geq 2$, $\Omega(P) = Q$ et $S_P = (a, a - 1, b, b - 1)$, $a > b + 2$, or $S_P = (a, a - 1, a - 2)$. The largest part u of Q comes from a U -row C_a (type A), or C_b (type B) or C_{a-1} (type C).

Example

Type A: $P = (\underbrace{5, 4}, 2, 1)$. Type B: $P = (5, 4, \underbrace{2, 2, 2})$. $|C_2| = 10$

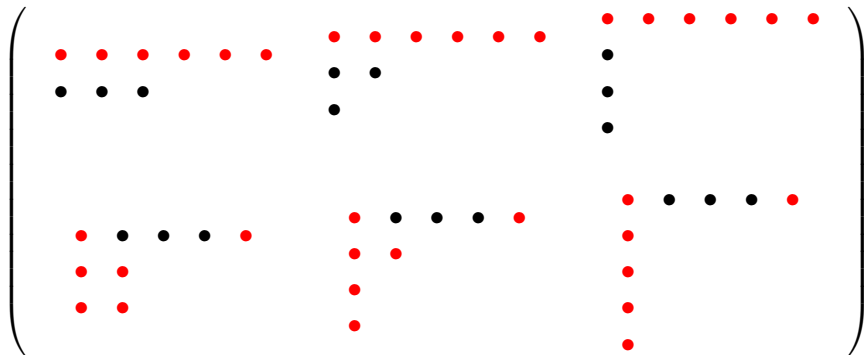
Type C: $P = (5, \underbrace{4, 4, 4, 3, 3}, 2)$, $|C_4| = 20$

Theorem ([Obl2, Z] Special $\Omega^{-1}(u, u - r)$)

The table conjecture Ω^{-1} is shown for $2 \leq r \leq 4$ (P. Oblak); and also for $u \gg r$ – the “normal pattern” case when each A row is followed immediately by a B hook (R.Zhao).

Example (Normal pattern)

The table $\mathcal{T}(Q)$ for $Q = (6, 3)$ has “normal pattern”: the first row $(6, 3), (6, [3]^2), (6, [3]^3)$ is type A, the second $(5, [4]^2), (5, [4]^3), (5, [4]^4)$ is a hook of type B.



Example ($\mathcal{T}(Q)$ for $Q = (12, 3)$, First $C \setminus A \cup B$ case $[Z]$.)

$\mathcal{T}(Q)$	3	$[3]^2$	$[3]^3$
8	$(12, 3)$	$(12, [3]^2)$	$(12, [3]^3)$
$[8]^2$	$([12]^2, 3)$	$[12]^2, [3]^2)$	$([12]^2, [3]^3)$
$[8]^3$	$(5, [10]^3)$	$(5, [10]^4)$	$(5, [10]^5)$
$[8]^4$	$([12]^3, [3]^2)$	$([12]^3, [3]^3)$	$(5, [10]^6)$
$[8]^5$	$(4, [10]^4, 1)^C$	$([7]^2, [8]^5)$	$(5, [10]^7)$
$[8]^6$	$([12]^4, [3]^3)$	$([7]^2, [8]^6)$	$(5, [10]^8)$
$[8]^7$	$([9]^3, [6]^5)$	$([7]^2, [8]^7)$	$(5, [10]^9)$
$[8]^8$	$([9]^3, [6]^6)$	$([7]^2, [8]^8)$	$(5, [10]^{10})$

Combinatorial Relation between $\mathcal{T}(Q)$ and Durfee squaresDefinition ($\mathcal{DH}(Q)$, Q stable)

Let Q be a stable partition. Denote by $\mathcal{DH}(Q)$ the set of all partitions having diagonal hook lengths Q .

Example ($\mathcal{DH}(Q)$ for $Q = (6, 3)$)

The inside diagonal hook h_{22} has length 3 so can be

$$P' = (3) \bullet \bullet \bullet, (2, 1) \begin{array}{c} \bullet \\ \bullet \end{array} \bullet, \text{ or } (1, 1, 1) \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array} .$$

Corollary (Bijection $\mathcal{T}(Q)$ et $\mathcal{DH}(Q)$.)

Let $Q = (u, u - r)$. There is a bijection $\theta : \mathcal{T}(Q) \rightarrow \mathcal{DH}(Q)$ that preserved the number of parts of P .

Proof. It is simple to write the tables $\mathcal{DH}(Q)$ by adding longer diagonal hooks; so it is easy to count $|\mathcal{DH}(Q)|$. It's the same number for $\mathcal{T}(Q)$ after the Theorem. We take $\theta(\mathcal{T}_{ij}(Q)) = \mathcal{DH}_{i,j}(Q)$.

Question

Can we define θ^{-1} combinatorially? A “jeu de taquin”?

If we can extend the definition of θ to Q with $k > 2$ parts, this can help construct the tables $\mathcal{T}(Q)$, as $\mathcal{DH}(Q)$ is easy to write down.

Example (θ for $Q = (6, 3)$)

The map $\theta(\mathcal{T}(Q)_{ij}) = \mathcal{DH}(Q)_{ij}$. Here

$$\mathcal{T}(Q) = \begin{pmatrix} (6, 3) & (6, [3]^2) & (6, [3]^3) \\ (5, [4]^2) & (5, [4]^3) & (5, [4]^4) \end{pmatrix}.$$

$$\mathcal{DH}(Q) = \begin{pmatrix} (5, 4) & (4, 3, 2) & (3, 2, 2, 2) \\ (4, 4, 1) & (3, 3, 2, 1) & (2, 2, 2, 1) \end{pmatrix}.$$

Example (Case $r_P = 1$, $\text{dh}(P)$ has 1×1 Durfee square.)

Soit $n = 5$, $Q = (5)$.

$$\mathcal{T}(Q) = ([5], [5]^2, [5]^3, [5]^4, [5]^5)$$

$$\mathcal{DH}(Q) = ((5), (4, 1), (3, 1^2), (2, 1^4), (1^5)) \text{ (single diagonal hook).}$$

Lemma

GF for $\#P \vdash n \mid \text{durf}(Q) = 2$ is shifted <http://oeis.org/A006918>.

$$\text{durf}_2(n) = \begin{cases} \frac{(n+1)(n-1)(n-3)}{24} & \text{if } i \geq 5 \text{ is odd} \\ \frac{1}{4} \binom{n}{3} & \text{if } i \geq 4 \text{ is even,} \end{cases} \quad (1)$$

Definition (Key S_Q of a stable Q)

Let $Q = (q_1, q_2, \dots, q_k)$, $q_i \geq q_{i+1} + 2$, $1 \leq i < k$ be stable. The key $S_Q = (q_1 - q_2 - 1, q_2 - q_3 - 1, \dots, q_{k-1} - q_k - 1, q_k)$.

Example

For $Q = (u, u - r)$ the key is $S_Q = (r - 1, u - r)$.

For $Q = (12, 6, 2)$ the key is $S_Q = (5, 3, 2)$

Corollary of the box conjecture

For Q RR, there is an isomorphism $\theta : \Omega^{-1}(Q) \rightarrow \mathcal{DH}(Q)$, that preserves numbers of parts.

Problems Find θ explicitly.

Give the table $\mathcal{T}(Q)$. (Find “hooks” for $k \geq 3$.)

Box conjecture for $\Omega^{-1}(Q)$

Let $Q = (q_1, \dots, q_k)$ be stable of key S_Q . Then

- (i) The partitions $\Omega^{-1}(Q)$ form a k -box $\mathcal{T}(Q)$ such that $\mathcal{T}(Q)_I, I = (i_1, \dots, i_k)$ has $|I|$ parts.
- (ii) The codimension of the similarity orbit of $\mathcal{T}(Q)_I$ in \mathcal{N}_Q is $|I| - k$.

Example ($S_Q = (2, 2, 2)$)

Take $Q = (8, 5, 2)$ so $S_Q = (2, 2, 2)$.

The two floors of $\mathcal{T}(Q)$ are

$$\begin{pmatrix} (8, 5, 2) & (8, 5, 1^2) \\ (8, 4, 2, 1) & (8, 4, 1^3) \end{pmatrix}, \begin{pmatrix} (7, 4, 2^2) & (7, 4, 2, 1^2) \\ (7, 3^2, 1^2) & (7, 4, 1^4) \end{pmatrix}.$$

The corresponding floors of $\mathcal{DH}(Q) = \theta(\mathcal{T}(Q))$ are

$$\begin{pmatrix} (6, 5, 4) & (5, 4, 3^2) \\ (5, 4^2, 2) & (4, 3^3, 2) \end{pmatrix}, \begin{pmatrix} (5^2, 4, 1) & (4^2, 3^2, 1) \\ (4^3, 2, 1) & (3^4, 2, 1) \end{pmatrix}.$$

Question: Can we explain these results? *Not yet!*

Lie algebra perspective:

The columns of $\mathcal{D}(P)$ are weight spaces for the sl_2 triple of B . But the S_n irreps for $P \in \mathcal{T}(Q)$ and $\theta(P) \in \mathcal{DH}(Q)$ have different VS dimensions.

Map to the Hilbert scheme:

Let $B = J_Q$. The map

$$\pi : \mathcal{N}_B \rightarrow \text{Hilb}^n \mathbb{k}[x, y]: A \rightarrow \mathbb{k}[A, B]$$

defines an image, whose fixed points under a \mathbb{C}^* action correspond to the monomial ideals of $\mathcal{T}(Q)$, so to homology classes on $\pi(\mathcal{N}_B)$. Will this explain the codimensions in $\mathcal{T}(Q)$?

Combinatorial questions arising from $P \rightarrow \Omega(P)$.

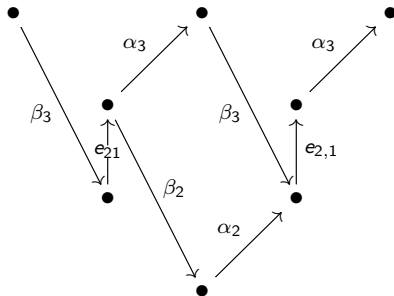
- (a) Poset $\mathcal{D}(P)$: Is $\lambda(\mathcal{D}_P) = \lambda_U(\mathcal{D}_P)$?
- (b) Explain the map $\theta^{-1} : \mathcal{DH}(Q) \rightarrow \mathcal{T}(Q)$ combinatorially.
- (c) Verify $\#\{P \vdash n \text{ with } p \text{ parts and } r_P = k\}$ is the expected sum.
- (d) An a -cluster is a partition $P = (p_1 \geq \dots \geq p_t)$ with $p_1 - p_t \leq a$.
 $r_{a,P} = \min\{\# \text{ } a\text{-clusters needed to cover } P\}$.
 $V_{a,k}(n) = \{P \vdash n \mid r_{a,P} = k\}$.
 Determine $|V_{a,k}(n)|$.
- (e) Consider other posets \mathcal{P} with multiplicities, and a linear action $B \rightarrow$ on vertices(\mathcal{P}). Consider $A \in \mathfrak{J}(\mathcal{P})$ commuting with B .
 Is $\lambda(\mathcal{P}) = \lambda^B(\mathcal{P})$?

Jordan type of modules over $A = k[x, y]/(x^p, y^p)$

Acknowledgment

We appreciate discussions with and helpful comments by Don King, Alfred Noel, George McNinch, and a conversation of Rui and Tony with Barry Mazur. We are grateful for the insights of P. Oblak, T. Košir and others who contributed questions and results that have been important to our work. William Keith responded to our query about $\mathcal{DH}(Q)$. We appreciate use of notes of Rick Porter on LaTeX, xy-pic, and his advice.

Thank you for your attention and questions!

Appendix: $\Omega(P)$ and its smallest part (Survey)Figure : Diagram of the poset \mathcal{D}_P : $P = (3, 2, 2, 1)$.

Def. (U -chain)

A U -chain C_i in \mathcal{D}_P is the saturated (maximal) chain through the union of three subsets of vertices:

- (i) All rows of length $i, i - 1$, corresponding to an AR subpartition of P .
- (ii) A descending chain from the source – the top left vertex of \mathcal{D}_P – to the vertex at the start of the lowest length- i row.
- (iii) An ascending chain from the vertex at the end of the highest length- i row to the sink – the top right vertex of \mathcal{D}_P .

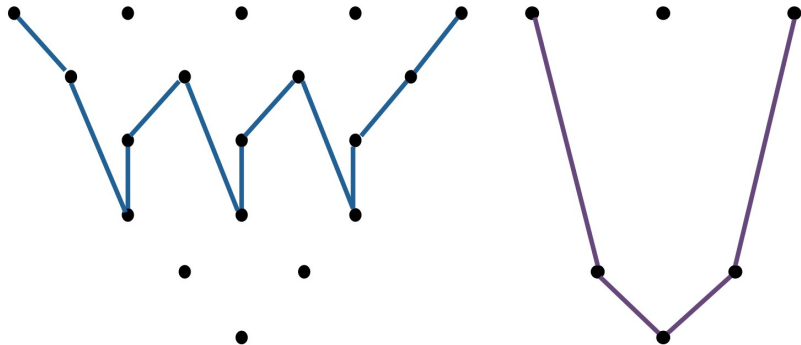


Figure : U -chain C_4 for $P = (5, 4, 3, 3, 2, 1)$ and new U -chain for $P' = (3, 2, 1)$. [Source: LK NU GASC talk 2013]

Oblak Recursive Conjecture

We obtain $\Omega(P)$ as follows from \mathcal{D}_P :

- (i) Choose a maximum length U -chain in \mathcal{D}_P . Its length is q_1 , the largest part of $\Omega(P)$.
- (ii) Remove the vertices in the chain from \mathcal{D}_P , obtaining a smaller partition P' . If $P' \neq \emptyset$ then $\Omega(P) = (q_1, \Omega(P'))$ (go to (i)).

Warning. The poset $\mathcal{D}_{P'}$ in the Oblak recursion is *not* in general a subposet of \mathcal{D}_P .

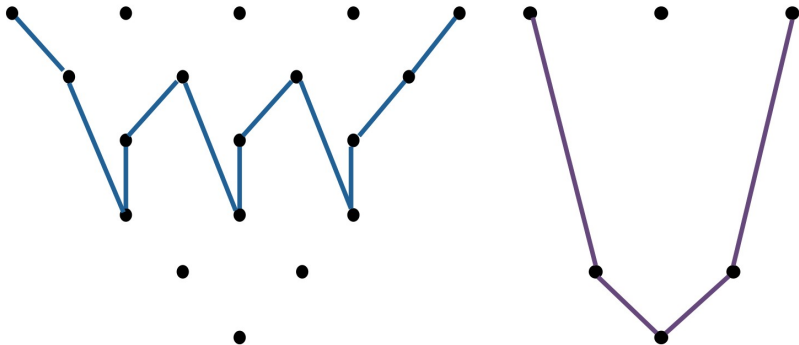


Figure : U -chain for $P = (5, 4, 3, 3, 2, 1)$ and new U -chain for $P' = (3, 2, 1)$. So $\Omega(P) = (12, 5, 1)$.

Theorem (P. Oblak [Obl1] – Index of $\mathfrak{Q}(P)$)

The index (largest part) of $\mathfrak{Q}(P)$ is the length of the longest U -chain in \mathcal{D}_P .

Theorem (L. Khatami [Kh1] – $Ob(P) = \lambda_U(\mathcal{D}_P)$)

The partition $Ob(P)$ obtained by the Oblak recursive process is independent of the choices of AR subpartitions; and $Ob(P) = \lambda_U(\mathcal{D}_P)$, obtained as $\lambda(\mathcal{D}_P)$ below by restricting to sets of U -chains.

Definition

$P \geq P'$ in the orbit closure (Bruhat) order if

$$\forall i \sum_{u=1}^i p_u \geq \sum_{u=1}^i p'_u.$$

Theorem (I.L.Khatami [IKh])

$$\Omega(P) \geq Ob(P).$$

Proof idea. For each maximal-length set of s U -chains, we specify an $A \in \mathcal{N}_B$ such that $\dim_k k[A] \circ \{v_1, \dots, v_s\}$ where the v_i are initial elements, agrees with the sum of the first s parts of $Ob(P)$. This involves an analysis of the sets of chains from the v_i to all the vertices covered by the s U -chains.

Def. (C. Greene et al, see[BrFo])

Let \mathcal{D} be a poset without loops. Define $c_i = \max\#$ vertices covered by i chains. Set

$$\lambda(\mathcal{D}) = (c_1, c_2 - c_1, c_3 - c_2, \dots).$$

Theorem (C. Greene, S. Poljak, E.R. Gansner, see [BrFo])

Let \mathcal{D} be any finite poset without loops, and let A be a generic nilpotent matrix in the incidence algebra $\mathfrak{I}(\mathcal{D}_P)$. Then the Jordan type $P_A = \lambda(\mathcal{D})$.

Definition ([Kh1])

$\lambda_U(\mathcal{D}_P)$ is obtained by replacing arbitrary chains c_i in the definition of $\lambda(\mathcal{D}_P)$ by U -chains.

Question: Combinatorial Oblak conjecture

Is $\Omega(P) = \lambda(\mathcal{D}_P)$?

Since $Ob(P) = \lambda_U(\mathcal{D}_P) \geq \Omega(P)$ this is equivalent to

Is $\lambda_U(\mathcal{D}_P) = \lambda(\mathcal{D}_P)$?

The key issue is that $A \in \mathcal{N}_B$ commutes with B , that acts by moving vertices of \mathcal{D}_P to the right: this greatly restricts $A \in \mathcal{J}(\mathcal{D}_P)$. Does it matter for the Jordan type P_A ?

R. Basili answers “No” in her talk and ArXiv post that describe an apparent proof of the Oblak Recursive Conjecture.²

²R. Basili posted a preprint in June 2012 on her proof of the Oblak conjecture. It appears to implicitly assume $\text{char } k = 0$; The $\text{char } k = 0$ case by [IKh] implies the Oblak conjecture over any infinite field k .

Theorem (L. Khatami [Kh2] – Minimum part)

The minimum part of $\Omega(P)$ is a specified combinatorial invariant $\mu(P)$. Also

$$Ob(P)_{\min} = \Omega(P)_{\min} = \lambda(\mathcal{D}_P)_{\min} = \mu(P) \quad *$$

Proof idea. First show $\mu(P)$ is the minimum part of $\lambda_U(\mathcal{D}_P) = Ob(P)$. Then an intensive study of the antichains of \mathcal{D}_P shows $\lambda(\mathcal{D}_P)_{\min} = \mu(P)$. By [IKh], $Ob(P) \leq \Omega(P) \leq \lambda(\mathcal{D}_P)$, showing (*).

Corollary ([Obl1, KO, Kh2])

$\Omega(P)$ is explicitly known for $r_P \leq 3$, over any infinite field k .

The invariant $\mu(P)$ for a spread.

Let $P = ((p + s - 1)^{n_s}, \dots, p^{n_1})$ be an s -spread: $n_i > 0$ for $1 \leq i \leq s$. Set

$$\mu(P) = \min\{pn_{2i-1} + (p+1)n_{2j} \mid 1 \leq i \leq j \leq r_P\}$$

Note: For s odd $r_P = (s+1)/2$ so $n_{2r_P} = 0$ and $\mu(P) = p \cdot \min\{n_{2i-1} \mid 1 \leq i \leq r_P\}$.

Theorem (L.Khatami [Kh2])

For P a spread, $\mu(P)$ is the # of disjoint length- r_P antichains in \mathcal{D}_P .

Fact: [Gre, BrFo] $\lambda(\mathcal{D}_P)_{\min} = \#$ length r_P anti-chains in \mathcal{D}_P .

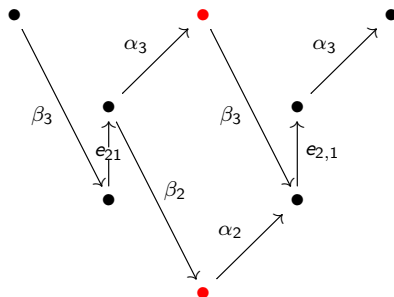


Figure : $\mu(P) = 1$ for $P = (3, 2, 2, 1)$, $Q(P) = (7, 1)$

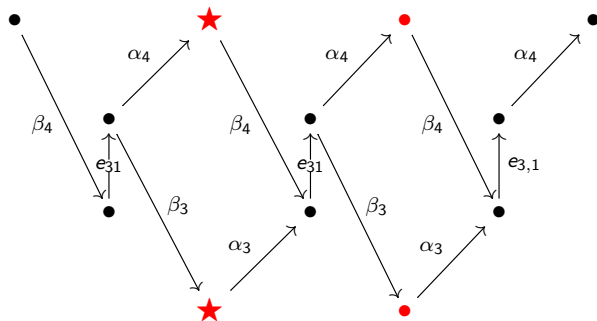






















Figure : $\mu(P) = 2$ for $P = (4, 3, 3, 2)$, $\Omega(P) = (10, 2)$






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