Types de Jordan de deux matrices nilpotentes qui commutent

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Résumé

La classe de similitude d'une matrice $n \times n$ nilpotente B sur un corps k est donnée par une partition P_B de n donnant les blocs de Jordan : c'est le "type de Jordan" de B. Nous étudions les paires de matrices nilpotentes A, B qui commutent : $A \in N(B)$ le commutant nilpotent de B. Étant donné $P = P_B$, nous écrivons $Q = \mathfrak{Q}(P)$ pour le type de Jordan d'une matrice A générique dans $N(B)$. P. Oblak, T. Košir, R. Basili, L. Khatami, D. Panyushev et d'autres ont étudié l'application $P \rightarrow \mathfrak{Q}(P)$: $\mathfrak{Q}(P)$ est une partition "stable" dont deux composantes quelconques diffèrent d'au moins deux unités. Si $Q = (q_1, q_2, \ldots, q_r)$ est stable, nous conjecturons qu'il y a une boîte $\mathcal{T}(Q)$ du dimension r, dont les entrées sont des partitions P telles que $\mathfrak{Q}(P) = Q$. Nous le démontrons quand $r = 2$, où $\mathcal{T}(Q)$ est un tableau rectangulaire des partitions. Nous étudions aussi les équations de l'orbite de telle partition dans $N(B)$ quand $r = 2$.

Travail en commun avec Leila Khatami, Bart Van Steirteghem, Rui Zhao et aussi pour les équations d'orbites avec Mats Boij.

Section 0 : Les algèbres Artiniens Gorensteins $\mathcal{A}$, quotients de $\mathsf{k}\{x, y\}$, et deux matrices nilpotentes.

When the Hilbert function H of an Artinian algebra $\mathcal{A}$ is fixed, there is an upper bound on the sequences that can occur as Jordan type for multiplication m_ℓ by $\ell = ax + by \in \mathcal{A}_1$ on $\mathcal{A}$. Our work pertains to the set of Jordan types of m_ℓ, given that $\mathcal{A}$ is an Artinian Gorenstein quotient of $\mathsf{k}\{x, y\}$.

Example

Let $\mathcal{A} = \mathsf{k}\{x, y\}/I, I = (xy, y^2 + x^3) = f^\perp$ where $f = y^2 - x^3$. Here $H(\mathcal{A}) = (1, 2, 1, 1)$. The Jordan type of the multiplication m_x by $x \in \mathcal{A}$ is the conjugate partition$H^\vee = (4, 1)$ to H.
Question. What. are the possible Jordan types P_ℓ of $m_\ell.\ell \in \mathcal{A}$?
Refined question Assume that $(4, 1)$ is the *maximum* Jordan type of a matrix commuting with m_ℓ. What are the possible Jordan types P_ℓ?

Answer

. Besides $(4,1)$, here $P=(3,1,1)$ is the only other partition for which $(4,1)$ is the maximum commuting Jordan type (in the Bruhat order). Other partitions of 5, as $(3,2)$, commute with $[5]$.

Theorem

Let $Q=(u,u-r), u>r\geq 2$. There are exactly $(r-1)(u-r)$ partitions such that $\mathfrak{Q}(P_{ij})=Q$, They can be arranged in a table $\mathcal{T}(Q)=\{P_{ij}(Q), 1\leq i\leq r-1, 1\leq j\leq u-r\}$ such that P_{ij} has $i+j$ parts.

Recall, $\mathcal{N}_B$ denotes the irreducible family of nilpotent matrices commuting with B. When $B=J_Q$, $\mathcal{N}_B$ is an affine space. We denote by $\mathfrak{Z}(P)\subset\mathcal{N}_B$ the locus of all $A\in\mathcal{N}_B$ such that $P_A=P$.

Conjecture (with M. Boij) The locus $\mathfrak{Z}(P_{ij})$ in $\mathcal{N}_B$, is a CI defined by a specified set of $i+j-2$ linear and quadratic equations.

Example

$$\mathcal{T}([5]) = \begin{pmatrix} [5] & (3,2) & (2,2,1) & (2,1,1,1) & (1,1,1,1,1) \end{pmatrix}.$$

$$\mathcal{T}((4,1)) = \begin{pmatrix} (4,1) \\ (3,1,1) \end{pmatrix};$$

$$\mathcal{T}(4,2)) = \begin{pmatrix} (4,2) & (4,1,1) \end{pmatrix};$$

$$\mathcal{T}(5,3) = \begin{pmatrix} (5,3) & (5,2,1) & (5,1,1,1) \end{pmatrix};$$

$$\mathcal{T}(5,2) = \begin{pmatrix} (5,2) & (5,1,1) \\ (4,2,1) & (4,1,1,1) \end{pmatrix};$$

$$\mathcal{T}(6,3) = \begin{pmatrix} (6,3) & (6,2,1) & (6,1,1,1) \\ (5,2,2) & (5,2,1,1) & (5,1,1,1) \end{pmatrix}.$$

Section 1 : L'application $\mathfrak{Q} : P \to \mathfrak{Q}(P)$

Definition (Nilpotent commutator $\mathcal{N}_B$)

$V \cong \mathsf{k}^n$ vector space over an infinite field k.
$A, B \in \mathrm{Mat}_n(\mathsf{k}) = \mathrm{Hom}_{\mathsf{k}}(\mathrm{V}, \mathrm{V})$, We assume $A^n = B^n = 0$.

$P \vdash n$ partition of n ;
$J_P =$ Jordan block matrix of Jordan type P

$\mathcal{C}_B \subset \mathrm{Mat}_n(\mathsf{k})$ centralizer of B.

$\mathcal{N}_B \subset \mathcal{C}_B$: the variety of nilpotent elements of $\mathcal{C}_B$.

$P_A =$ Jordan type of A : the partition giving the sizes of the Jordan blocks of a Jordan block matrix CAC^{-1} similar to A.

Fact : $\mathcal{N}_B$ is an irreducible variety [Bas1, BI].

Def : $\mathfrak{Q}(P) = P_A$ for A generic in $\mathcal{N}_B, B = J_P$.

Problem 1. Given the partition P, determine $\mathfrak{Q}(P)$

Fact. $\mathfrak{Q}(P)$ is Rogers-Ramanujan (RR) : the parts of $\mathfrak{Q}(P)$ differ by at least two (T. Košir and P. Oblak).

Problem 2. Given the RR partition Q determine $\mathfrak{Q}^{-1}(Q)$.

Status : Prob. 1 : P. Oblak (2008) made a Recursive Conjecture for $\mathfrak{Q}(P)$. Work by P. Oblak-T.Košir, L. Khatami, I-Khatami, R. Basili. Known for Q with $k \leq 3$ parts (P. Oblak, L. Khatami). "Half" is shown in [IK] ; R. Basili proposed a proof in 2014.

Prob 2 : We show the Table conjecture of P. Oblak and R. Zhao for $Q = (u, u - r), u > r > 1$. Our Box conjecture for $\mathfrak{Q}^{-1}(Q)$ is open for RR partitions Q with $k > 2$ parts..

Classical problems ! Canonical form is due to C. Jordan, 1870. But the map $P \to \mathfrak{Q}(P)$ was not studied classically.[1]

In 2006, three independent groups began to work on the $P \to \mathfrak{Q}(P)$ problem

P. Oblak and T. Košir (Ljubljana)

D. Panyushev (Moscow)

R. Basili, I.-, and L. Khatami (Perugia, Boston).

Connected to Hilbert scheme work of J. Briançon, M. Granger, R. Basili, V. Baranovsky, A. Premet, N. Ngo and K. Šivic.

Links to work of E. Friedlander, J. Pevtsova, A. Suslin, on representations of Abelian p-groups [FrPS,CFrP].

1. Instead, I. Schur (1905), N.Jordan, M. Gerstenhaber (1958), E. Wang (1979) studied *maximum dimension* commuting subalgebras/nilpotent subalgebras of $\mathcal{C}_B, \mathcal{N}_B$.

Lemma

Let $B = J_n$, the regular partition of n, a single Jordan block. Then $A \in \mathcal{C}_B$ if and only if A is a polynomial in B.
Let $p(X) = X^k \cdot u(X), u(X)$ a unit in $k[X]/(X^n)$, and let $A = p(B)$. Then the Jordan type $P_A = P_{B^k}$.

Example

For $n = 4, B = \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}, A = \begin{matrix} \epsilon & x_a & x_b & x_c \\ 0 & \epsilon & x_a & x_b \\ 0 & 0 & \epsilon & x_a \\ 0 & 0 & 0 & \epsilon \end{matrix} \in \mathcal{C}_B$.

So $A = \epsilon I + x_a B + x_b B^2 + x_c B^3$.
Take $k = 2$. Then $\epsilon = x_a = 0, x_b \neq 0, p(X) = X^2 \cdot (x_b + x_c X)$, and the Jordan type of A is that of $B^2 = \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$. $P_{B^2} = (2, 2)$.

Definition (Almost rectangular)

Let $B = J_{(n)}$, a regular Jordan block, and denote by $[n]^k = P_{B^k}$.

Lemma

For $n = kq$, $[n]^k = (q^k) = (q, q, \ldots, q)$.
For $n = kq + r, 0 < r < k$, $[n]^k = ((\lceil n/k \rceil)^r, (\lfloor n/k \rfloor)^{k-r})$

Proof

The number of parts of P_A is the rank of its kernel, so P_{B^k} has k parts. Since $(B^k)^{\lceil n/k \rceil} = 0$, each part of J_{B^k} is no greater than $\lceil n/k \rceil$.

We term $[n]^k$ *almost rectangular (AR)*. It is the unique partition of n that has k parts that differ at most by 1.

Example

$n = 5$,
$[5]^2 = (3, 2), [5]^3 = (2, 2, 1), [5]^4 = (2, 1, 1, 1), [5]^5 = (1, 1, 1, 1, 1)$.

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

B $\qquad$ B^2 $\qquad$ $J_{3,2}$

FIGURE : $B = J_{[5]}$, B^2, and $J_{3,2}$. Here $B^2 \sim J_{3,2}$.

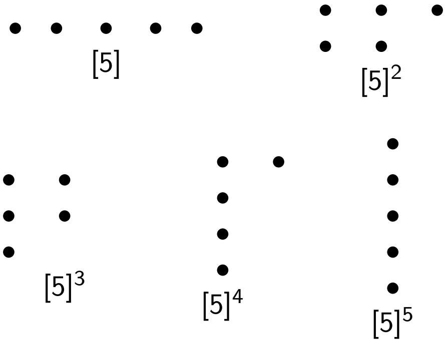


FIGURE : The AR partitions of 5.

Theorem ((R. Basili) $\mathfrak{Q}$ for $r_P = 1$)

For $P = [n]^k, \mathfrak{Q}(P) = [n]$ and $\mathfrak{Q}^{-1}([n]) = \{[n]^k, 1 \leq k \leq n\}$

Example

The Jordan type $P = (3,1)$ does not commute with (4).

Example ($\mathcal{U}_B$ for $B = J_P, P = (4)$)

$$P = (4), B = J_P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \mathcal{U}_B : A = \begin{pmatrix} 0 & x_a & x_b & x_c \\ 0 & 0 & x_a & x_b \\ 0 & 0 & 0 & x_a \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$A = x_a B + x_b B^2 + x_c B^3$, polynomial in B, so
$A = uB^k, k = 1,2,3,4, u$ unit in $\mathsf{k}[B]$

$P_A = [4]$, or $[4]^2 = (2,2)$ or $[4]^3 = (2,1,1)$ or $[4]^4 = (1,1,1,1)$

Theorem (R. Basili [Bas1])

$\mathfrak{Q}(P)$ *has* r_P *parts, where* $r_P = \min \#$ *AR partitions* P_i *such that* $P = \bigcup P_i$.

Theorem (R. Basili and I.- [BI])

$\mathfrak{Q}(P) = P \Leftrightarrow P$ *is RR : the parts of* P *differ pairwise by at least 2.*

Def. We call a $P \mid \mathfrak{Q}(P) = P$ "stable"

also "super-distinct" or "Rogers-Ramanujan" [AlBe, An].

Example

$$P = (\underbrace{3}, \underbrace{1}), \quad \mathfrak{Q}(P) = (3,1).$$

$$P = (\underbrace{5,4}, \underbrace{3,3,2}, \underbrace{1}), \quad \mathfrak{Q}(P) = (12,5,1).$$

Poset $\mathcal{D}_P$

Rows of vertices : Span the maximal irreducible B - invariant subspaces of V : each row corresponds to a part of P.
Arrows : non-zero elements in $A \in \mathcal{U}_B$ (max subalgebra of $\mathcal{N}_B$).

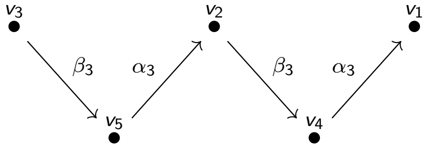


Diagram of $\mathcal{D}_P, P = (3, 2)$.

$$A = \left(\begin{array}{ccc|cc} 0 & x_{\alpha_3\beta_3} & x_{(\alpha_3\beta_3)^2} & x_{\alpha_3} & x_{\alpha_3\beta_3\alpha_3} \\ 0 & 0 & x_{\alpha_3\beta_3} & 0 & x_{\alpha_3} \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & x_{\beta_3} & x_{\beta_3\alpha_3\beta_3} & 0 & x_{\beta_3\alpha_3} \\ 0 & 0 & x_{\beta_3} & 0 & 0 \end{array}\right), v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

FIGURE : Generic element A of $\mathcal{U}_B, B = J_P$ where $P = (3, 2)$.

Details on the poset $\mathcal{D}_P$.

The vertices of $\mathcal{D}_P$ are a basis of V, arranged in rows spanning the cyclic B-spaces. B moves a vertex one step to the right.
Since A commutes with B, the action of A on any vertex follows from its action on the left-most vertex of $\mathcal{D}_P$ in the same row. This leads to the Toeplitz form of the matrix A in the basis $v_1, \ldots, v_5$.

Here $\mathcal{U}_B$ is an affine space with basis the 7 coordinates $x_{\alpha_3\beta_3}, \ldots, x_{\beta_3\alpha_3}$, each corresponding to a path in $\mathcal{D}_P$ from a left-most vertex in a row : for example $\alpha_3\beta_3$ is the path $v_3 \to v_5 \to v_2$ and $x_{\alpha_3\beta_3}$ may be regarded as the matrix with entries $A_{12} = A_{23} = x_{\alpha_3\beta_3}$ and otherwise zero.[2]
The *diagram* of $\mathcal{D}_P$ shows only the covering arrows. The poset $\mathcal{D}_P$ is sl_2 weighted, but does not have a height function in general (take $P = (4, 2, 1)$).

2. For $B = J_P$, $\mathcal{U}_B = \mathcal{N}_B$ when P has distinct parts ; otherwise we take $A \in \mathcal{U}_B$ to be strictly upper triangular on the subspaces spanned by the leftmost vertices of equal-length rows : see any of [Kh1, BIK, IKh, IKVZ].

The simplest poset, $\mathcal{D}_P$ for $P = (5)$.

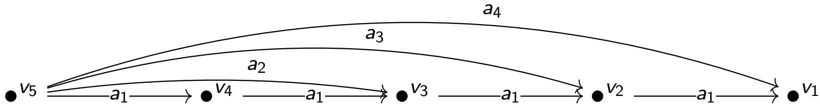


FIGURE : Diagram of $\mathcal{D}_Q$ and maps for $Q = (5)$.

Example

When [3] $a_1 = 0, a_2 \neq 0$ then we have strings (cyclic modules) $v_5 \to v_3 \to v_1$ and $v_4 \to v_2$ so $P_A = (3,2) = [5]^2$.

When $a_1 = a_2 = 0, a_3 \neq 0$ then we have strings $v_5 \to v_2$ and $v_4 \to v_1$ and v_3, so $P_A = (2,2,1) = [5]^3$.

3. We write a_1 for x_{a_1}, ...

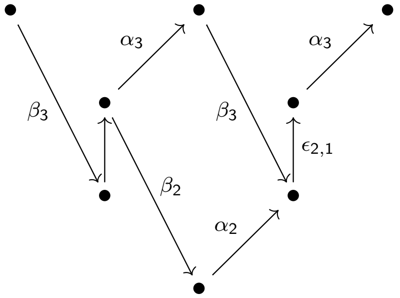


FIGURE : $\mathrm{Diag}(\mathcal{D}_P)$ for $P = (3, 2, 2, 1)$.

Theorem (P. Oblak and T.Košir [KO])

For $A \in \mathcal{N}_B$ generic, the Artinian algebra $\mathsf{k}[A, B]$ generated by A and B is Gorenstein, so it is a complete intersection (CI).

Proof. Uses an involution of the poset $\mathcal{D}_P$ of $\mathcal{N}_B$. See also [BIK, Thm. 2.20].

Corollary (ibid. vith F.H.S. Macaulay [Mac])

$\mathfrak{Q}(P)$ is stable ! ($\mathfrak{Q}(P)$ is RR : Parts differ pairwise by at least two). Also $\mathfrak{Q}(\mathfrak{Q}(P)) = \mathfrak{Q}(P)$.

Proof. After Macaulay, if $\mathcal{A}$ is CI, the jumps $e_i = H_i - H_{i+1}$ of $H = H(\mathcal{A})$ are each less or equal 1, which implies $H^\vee$ is RR.

Example

For $H = (1, 2, 3, 4, 3, 2, 2, 1)$, $H^\vee = (8, 6, 3, 1)$, which is RR.

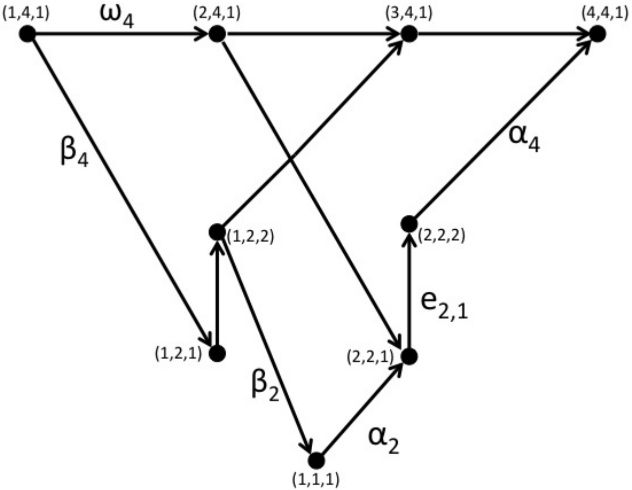


Diagram of the poset $\mathcal{D}_P$ and maps, $P = (4, 2, 2, 1)$.

Def : U-chain in $\mathcal{D}_P$ determined by an AR $P_1 \subset P$: a chain that includes all vertices of $\mathcal{D}_P$ from an AR subpartition P_1, + two tails.

The first tail descends from the source of $\mathcal{D}_P$ to the AR chain of P_1, and the second tail ascends from the AR chain to the sink of $\mathcal{D}_P$.

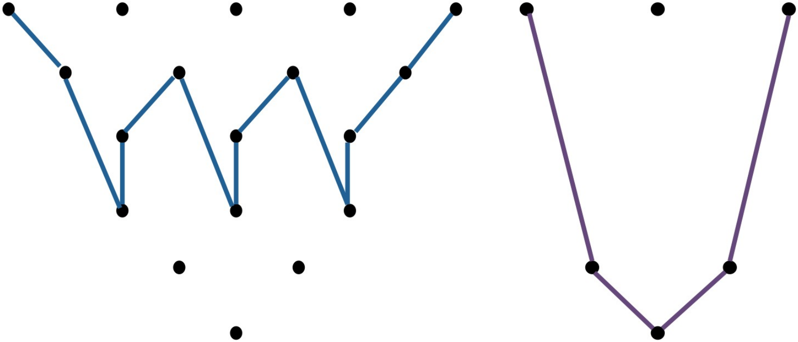


FIGURE : U-chain C_4 : $P = (5,4,3,3,2,1)$ and new U-chain of $P' = P - C_4 = (3,2,1)$.[from LK NU GASC talk 2013]

Oblak Recursive Conjecture

One obtains $\mathfrak{Q}(P)$ from $\mathcal{D}_P$:

(i) Let C be a longest U-chain of $\mathcal{D}_P$. Then $|C| = q_1$, the biggest part of $\mathfrak{Q}(P)$.

(ii) Remove the vertices of C from $\mathcal{D}_P$, giving a partition $P' = P - C$. If $P' \neq \emptyset$ then $\mathfrak{Q}(P) = (q_1, \mathfrak{Q}(P'))$ (Go to (i).).

Warning ! The poset $\mathcal{D}_{P'}$ is not a subposet of $\mathcal{D}_P$.

Theorem (P. Oblak[Obl1] – Index of $\mathfrak{Q}(P)$)

The index of $\mathfrak{Q}(P) =$ is the length of the longest U-chain C of $\mathcal{D}_P$.

Theorem (L. Khatami [Kh1] – $Ob(P) = \lambda_U(\mathcal{D}_P)$)

The partition $Ob(P)$ obtained by Oblak recursion is independent of choices of AR subpartitions, and $Ob(P) = \lambda_U(\mathcal{D}_P)$, obtained in the same way as $\lambda(\mathcal{D}_P)$ but using U-chains. [4]

Work of L. Khatami (smallest part of $\mathfrak{Q}(P)$) and P. Oblak (index) show the ORC for $r_P \leq 3$. Work of I-L. Khatami (1/2 Oblak Rec Conj), and a proposed proof of R. Basili (Oblak Rec Conj for $\mathrm{char}\ k = 0$, 2014) appear to show the Recursive Conjecture.

Is there another proof ? By [I-Kh] it suffices to show $\lambda_U(\mathcal{D}_P) = \lambda(\mathcal{D}_P)$, a combinatorial statement about the poset $\mathcal{D}_P$.

4. A theory of E.R. Gansner, D. Kleitman, C. Greene, S. Poljak, T. Britz and S. Fomin assigns a partition $\lambda(\mathcal{P})$, using lengths of multichains of a poset $\mathcal{P}$.

Section 2 : Théorème de tableau : $\mathfrak{Q}^{-1}(Q)$ pour $Q=(u,u-r)$.

The set $\mathfrak{Q}^{-1}(Q)$ has been mysterious, even for $Q=(u,u-r), u>r>1$ where $P\to\mathfrak{Q}(P)$ is explicit. P. Oblak (2012) [Obl2] and R. Zhao (2013) proposed

Table theorem for $\mathfrak{Q}^{-1}(Q)$

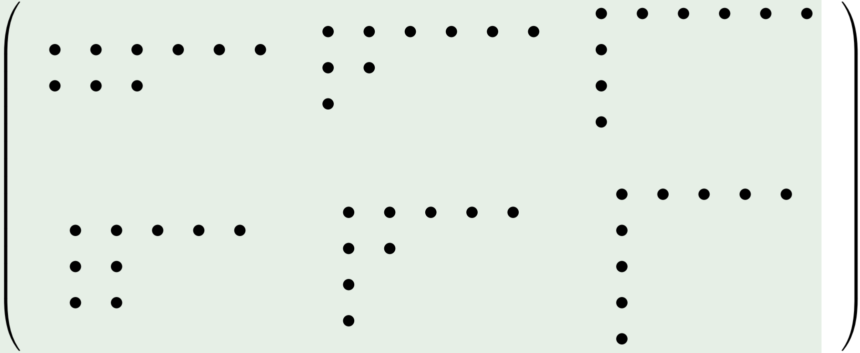
The elements of $\mathfrak{Q}^{-1}(Q), Q=(u,u-r), u>r>1$ form a $(r-1)\times(u-r)$ table $\mathcal{T}(Q)$ such that $\mathcal{T}(Q)_{i,j}$ has $i+j$ parts.

[P. Oblak : $\#\ \mathfrak{Q}^{-1}(Q)=(r-1)(u-r)$; R. Zhao : table $\mathcal{T}(Q)$].

Example (Table $\mathcal{T}(Q)$ for $Q = (6,3)$)

Let $Q = (6,3)$.

$$\mathcal{T}(Q) = \begin{pmatrix} (6,3) & (6,[3]^2) & (6,[3]^3) \\ (5,[4]^2) & (5,[4]^3) & (5,[4[^4) \end{pmatrix} \text{ of types } \begin{pmatrix} A & A & A \\ B & B & B \end{pmatrix}$$



Definition (Type A,B,C partitions in $\mathfrak{Q}^{-1}(Q)$)

Let $Q = (u, u-r), u > r > 2$, $\mathfrak{Q}(P) = Q$ et $S_P = (a, a-1, b, b-1)$, $a > b+2$, or $S_P = (a, a-1, a-2)$. The largest part u of Q comes from a U-row C_a (type A), or C_b (type B) or C_{a-1} (type C).

Example

Type A : $P = (\underbrace{5, 4}, 2, 1)$. Type B : $P = (5, 4, \underbrace{2, 2, 2})$. $|C_2| = 10$

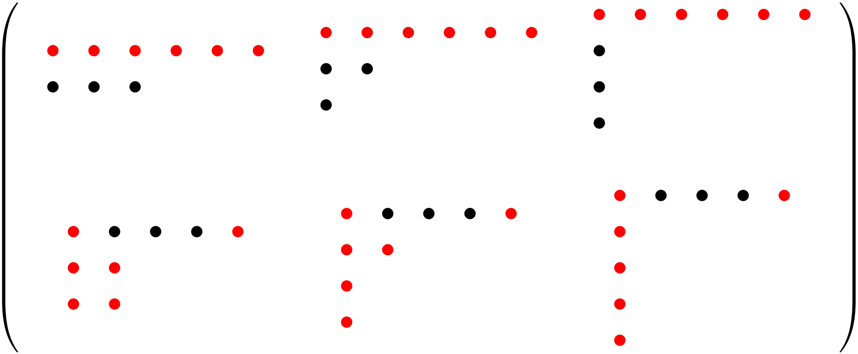
Type C : $P = (5, \underbrace{4, 4, 4, 3, 3}, 2)$, $|C_4| = 20$

Theorem ([Obl2, Z] Special cases of $\mathfrak{Q}^{-1}(u, u-r)$)

The table theorem was shown for $2 \leq r \leq 4$ (P. Oblak); and also for $u >> r$ – the "normal pattern" case when each A row is followed immediately by a B hook (R. Zhao).

Example (Normal pattern)

The table $\mathcal{T}(Q)$ for $Q = (6, 3)$ has "normal pattern" : the first row $(6, 3), (6, [3]^2), (6, [3]^3)$ is type A ; the second row $(5, [4]^2), (5, [4]^3), (5, [4]^4)$ is a "hook" of type B.



Theorem ([IKVZ] Table $\mathcal{T}(Q)$)

Let $Q = (u, u - r)$. There is an $(r-1) \times (u-r)$ table $\mathcal{T}(Q)$ comprised of all the partitions from $\mathfrak{Q}^{-1}(Q)$, arranged in specified rows of type A, and hooks whose partitions have type B or C,B. The entry $P_{ij}(Q)$ in the i-row, j-column of $\mathcal{T}(Q)$ has $i + j$ parts.

$\mathcal{T}(Q)$ contains all the set $\mathfrak{Q}^{-1}(Q)$.

Properties : There are $\min\{u - r, \lfloor \frac{r-1}{2} \rfloor\}$ type B or C,B hooks. They fit together with the type A partial rows as in a puzzle.

Example ($\mathcal{T}(Q)$ for $Q = (8,3)$, normal pattern)

$$\mathfrak{Q}^{-1}(8,3) = \begin{pmatrix} (8,3) & (8,[3]^2) & (8,[3]^3) \\ \mathbf{(5,[6]^2)} & \mathbf{(5,[6]^3)} & \mathbf{(5,[6]^4)} \\ ([8]^2,[3]^2) & ([8]^2,[3]^3) & \mathbf{(5,[6]^5)} \\ ([7]^2,[4]^3) & ([7]^2,[4]^4) & \mathbf{(5,[6]^6)} \end{pmatrix}$$

of types $\begin{pmatrix} A & A & A \\ B & B & B \\ A & A & B \\ B' & B' & B \end{pmatrix}$. Note two B "hooks".

Example ($\mathcal{T}(Q)$ for $Q = (12, 3)$, First C\ A∪ B case [Z].)

$\mathcal{T}(Q)$	3	$[3]^2$	$[3]^3$
8	$(12, 3)$	$(12, [3]^2)$	$(12, [3]^3)$
$[8]^2$	$([12]^2, 3)$	$[12]^2, [3]^2)$	$([12]^2, [3]^3)$
$[8]^3$	$(5, [10]^3)$	$(5, [10]^4)$	$(5, [10]^5)$
$[8]^4$	$([12]^3, [3]^2)$	$([12]^3, [3]^3)$	$(5, [10]^6)$
$[8]^5$	$(4, [10]^4, 1)^C$	$([7]^2, [8]^5)$	$(5, [10]^7)$
$[8]^6$	$([12]^4, [3]^3)$	$([7]^2, [8]^6)$	$(5, [10]^8)$
$[8]^7$	$(\mathbf{[9]^3}, \mathbf{[6]^5})$	$([7]^2, [8]^7)$	$(5, [10]^9)$
$[8]^8$	$(\mathbf{[9]^3}, \mathbf{[6]^6})$	$([7]^2, [8]^8)$	$(5, [10]^{10})$

.

Idea of proof :

Use that we know $P \to \mathfrak{Q}(P)$ for Q with $r_P = 2$ parts (P. Oblak).

(i) Write out the A row, B/C hook decomposition of the table.

(ii) Specify the elements of $\mathcal{T}(Q)$, showing they are in $\mathfrak{Q}^{-1}(Q)$. ✓

(iii) Determine all the partitions P of type C having $\mathfrak{Q}(P) = Q$ and show that are in $\mathcal{T}(Q)$. This and the analogue for type A,B show that the table is the complete $\mathfrak{Q}^{-1}(Q)$. ✓

Table $\mathcal{E}(Q)$ of equations for loci $\mathfrak{Z}(P_{ij})$ in $\mathcal{N}_B$, $B = J_Q$.

Def : For $P \in \mathcal{T}(Q)$, the locus $\mathfrak{Z}(P) = \{A \in \mathcal{N}_Q \mid P_A = P\}$.

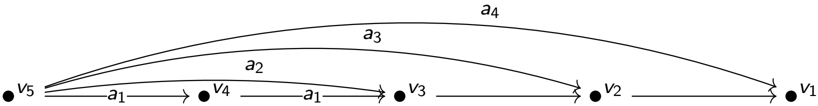


FIGURE : Diagram of $\mathcal{D}_Q$ and maps for $Q = (5)$.

Example (Equations for table loci $\mathcal{T}(Q)$, $Q = (5)$.)

$$\mathcal{T} = \big(5; \quad [5]^2; \quad [5]^3; \quad [5]^4; \quad [5]^5)\big)$$
$$\mathcal{E} = \big(-; \quad a_1; \quad a_1, a_2; \quad a_1, a_2, a_3; \quad a_1, a_2, a_3, a_4\big).$$

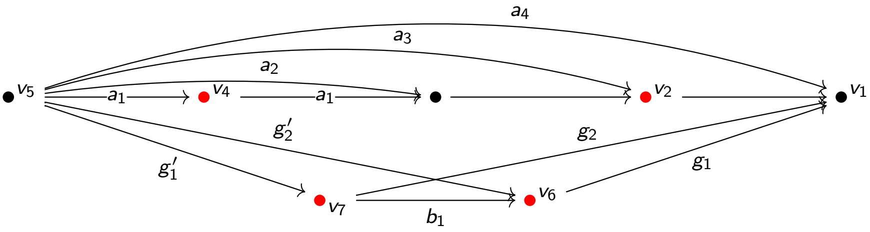


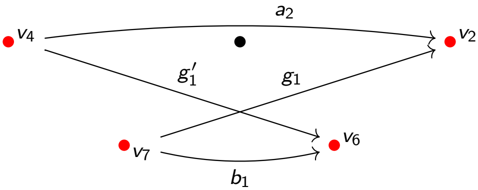
FIGURE : Diagram of $\mathcal{D}_Q$ and maps for $Q = (5, 2)$.

Example (Equations for table loci : $\mathcal{T}(Q), Q = (5, 2)$)

$$\mathcal{T} = \begin{pmatrix} (5,2) & (5,[2]^2) \\ (4,[3]^2) & (4,[3]^3) \end{pmatrix}$$

$$\mathcal{E} = \begin{pmatrix} - & b_1 \\ a_1 & a_1, |M| \end{pmatrix}, M = \begin{pmatrix} a_2 & g_1 \\ g_1' & b_1 \end{pmatrix}.$$

Subdiagram for $P = (5, 2)$ and matrix M



$P_A = (4, [3]^3)$ requires $a_1 = 0$ and

$\dim Ker(A) = 4 \Rightarrow \det M = 0$, $M = \begin{pmatrix} a_2 & g_1 \\ g_1' & b_1 \end{pmatrix}$.

Table Loci Conjecture (with M. Boij)

- (i) The locus $\mathfrak{Z}_{ij}$ in $\mathcal{U}_B$ of all matrices A of Jordan type $T_{ij}(Q), Q = (u, u-r), 1 \leq i \leq r-1, 1 \leq j \leq u-r$ is a complete intersection of codimension $i+j-2$ in $\mathcal{U}_B$.
- (ii) The locus $\mathfrak{Z}_{ij}$ is given by equations of which $\min\{i+j-2, r-2\}$ are linear, and $k = \max\{i+j-r, 0\}$ are quadratic $(k-1)$-th polarizations of a 2×2 determinant unique to the A-row or B hook.

Status of TLC : Work in progress with M. Boij, L. Khatami, Bart Van Steirteghem via image, kernel of $A \in \mathcal{U}_Q$.

An approach by T. Košir, K.Šivic, P. Oblak views the equations as polarizations of a determinant.

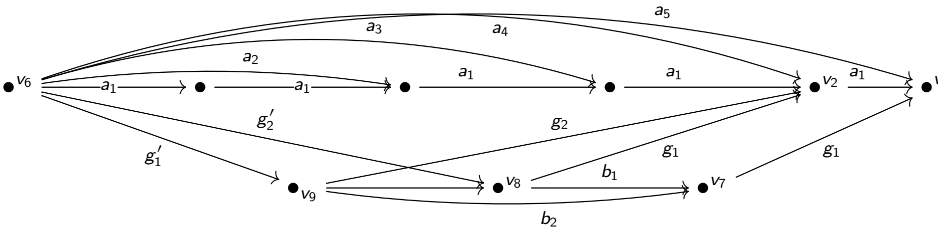


FIGURE : Diagram of $\mathcal{D}_Q$ and maps for $Q = (6, 3)$.

Coordinates in $\mathcal{U}_B$, $B = J_Q$, $Q = (6,3)$

$$A = \begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 & a_5 & g_1 & g_2 & g_3 \\ 0 & 0 & a_1 & a_2 & a_3 & a_4 & 0 & g_1 & g_2 \\ 0 & 0 & 0 & a_1 & a_2 & a_3 & 0 & 0 & g_1 \\ 0 & 0 & 0 & 0 & a_1 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g'_1 & g'_2 & g'_3 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & 0 & g'_1 & g'_2 & 0 & 0 & b_1 \\ 0 & 0 & 0 & 0 & 0 & g'_1 & 0 & 0 & 0 \end{pmatrix},$$

$$B : \begin{array}{c} a_1 = b_1 = 1 \\ \{a_2, \ldots a_5, g_1, \ldots, g_3, g'_1, \ldots, g'_3, b_2\} = 0. \end{array}.$$

Here $\mathcal{U}_B \cong \mathbb{A}^{13}$: the coordinates correspond to 8 paths in $\mathcal{D}_P$ from v_6 and 5 from v_9, where v_6, v_9 are the cyclic B-generators of V.

Example (Equations for table loci : $\mathcal{T}(Q), Q = (6,3)$)

$$\mathcal{T} = \begin{pmatrix} (6,3) & (6,[3]^2) & (6,[3]^3) \\ (5,[4]^2) & (5,[4]^3) & (5,[4]^4) \end{pmatrix}$$

$$\mathcal{E} = \begin{pmatrix} - & b_1 & b_1, |M_0| \\ a_1 & a_1, |M| & a_1, |M|, |N| = 0 \end{pmatrix}, \text{ where}$$

$$M_0 = \begin{pmatrix} a_1 & g_1 \\ g_1' & b_2 \end{pmatrix}, M = \begin{pmatrix} a_2 & g_1 \\ g_1' & b_1 \end{pmatrix}, N = \begin{pmatrix} a_3 & g_2 \\ a_2 & g_1 \\ g_2' & b_2 \\ g_1' & b_1 \end{pmatrix}.$$

$$|N| = 0 : a_3 b_1 - g_1' g_2 + a_2 b_2 - g_2' g_1 = 0.$$

$$\left(\det \begin{pmatrix} a_3 & g_2 \\ g_1' & b_1 \end{pmatrix} + \det \begin{pmatrix} a_2 & g_1 \\ g_2' & b_2 \end{pmatrix} = 0 \right)$$

Section 3 : Conjecture de Boite pour $\mathcal{T}(Q)$

.

Definition (Key S_Q of a stable Q)

Let $Q = (q_1, q_2, \ldots, q_k), q_i \geq q_{i+1} + 2, 1 \leq i < k$ be stable. The *key* $S_Q = (q_1 - q_2 - 1, q_2 - q_3 - 1, \ldots, q_{k-1} - q_k - 1, q_k)$.

Example

For $Q = (u, u - r)$ the key is $S_Q = (r - 1, u - r)$.
For $Q = (12, 6, 2)$ the key is $S_Q = (5, 3, 2)$

Box conjecture for $\mathfrak{Q}^{-1}(Q)$

Let $Q = (q_1, \dots, q_k)$ be stable of key S_Q. Then

(i) The partitions $\mathfrak{Q}^{-1}(Q)$ form a k-box $\mathcal{T}(Q)$ such that $\mathcal{T}(Q)_I, I = (i_1, \dots, i_k)$ has $|I|$ parts.

(ii) The codimension of the similarity orbit of $\mathcal{T}(Q)_I$ in $\mathcal{N}_Q$ is $|I| - k$.

(iii) The locus $\mathfrak{Z}(P_I)$ is an irreducible complete intersection defined by linear and quadratic equations in the variables of $\mathcal{U}_B, B = J_Q$.

Example ($S_Q = (2,2,2)$)

Take $Q = (8,5,2)$ so $S_Q = (2,2,2)$.
The two floors of $\mathcal{T}(Q)$ are

$$\begin{pmatrix} (8,5,2) & (8,5,1^2) \\ (8,4,2,1) & (8,4,1^3) \end{pmatrix}, \begin{pmatrix} (\mathbf{7},\mathbf{4},\mathbf{2^2}) & (7,4,2,1^2) \\ (7,3^2,1^2) & (7,4,1^4) \end{pmatrix}.$$

The corresponding floors of $\mathcal{DH}(Q) = \theta(\mathcal{T}(Q))$ are

$$\begin{pmatrix} (6,5,4) & (5,4,3^2) \\ (5,4^2,2) & (4,3^3,2) \end{pmatrix}, \begin{pmatrix} (\mathbf{5^2},\mathbf{4},\mathbf{1}) & (4^2,3^2,1) \\ (4^3,2,1) & (3^4,2,1) \end{pmatrix}.$$

Example ($S_Q = (3,3,3)$)

Take $Q = (11,7,3)$ so $S_Q = (3,3,3)$. See Figure.

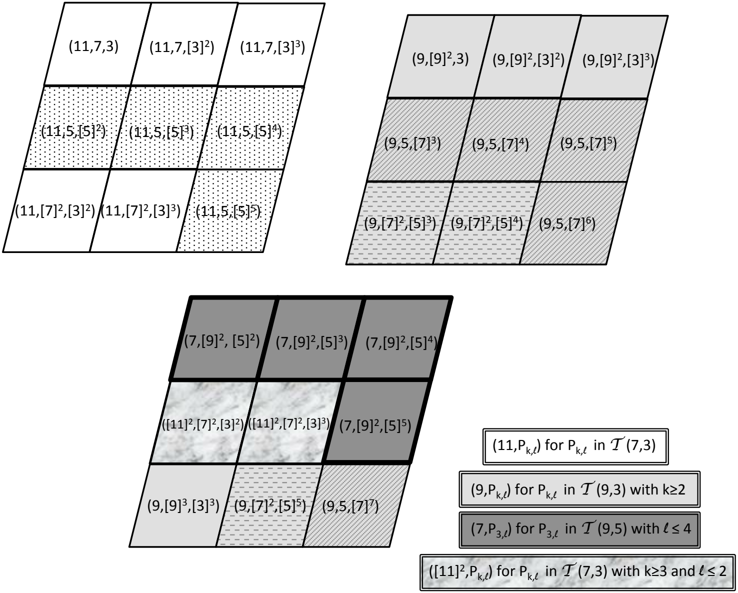


FIGURE : Box $\mathcal{B}(Q)$ for $Q=(11,7,3)$, $S_Q=(3,3,3)$.

Question : Can we explain these results ? *Not yet !*

Lie algebra perspective :

The columns of $\mathcal{D}(P)$ are weight spaces for the sl_2 triple of B. But the S_n irreps for $P \in \mathcal{T}(Q)$ and $\theta(P) \in \mathcal{DH}(Q)$ have different VS dimensions.

Map to the Hilbert scheme :

Let $B = J_Q$. The map

$$\pi : \mathcal{N}_B \to \mathrm{Hilb}^n \mathsf{k}[x, y] : A \to \mathsf{k}[A, B]$$

defines an image, whose fixed points under a $\mathbb{C}^*$ action correspond to the monomial ideals of $\mathcal{T}(Q)$, so to homology classes on $\pi(\mathcal{N}_B)$. Will this explain the codimensions in $\mathcal{T}(Q)$?

Combinatorial questions arising from $P \to \mathfrak{Q}(P)$.

(a) Poset $\mathcal{D}(P)$: Is $\lambda(\mathcal{D}_P) = \lambda_U(\mathcal{D}_P)$?

(b) Explain the map $\theta^{-1} : \mathcal{DH}(Q) \to \mathcal{T}(Q)$ combinatorially.

(c) Verify $\# \{P \vdash n$ with p parts and $r_P = k\}$ is the expected sum.

(d) An a-cluster is a partition $P = (p_1 \geq \ldots \geq p_t)$ with $p_1 - p_t \leq a$.
$r_{a,P} = \min\{ \#\ a$-clusters needed to cover $P\}$.
$V_{a,k}(n) = \{P \vdash n \mid r_{a,P} = k\}$.
Determine $|V_{a,k}(n)|$.
(e) Consider other posets $\mathcal{P}$ with multiplicities, and a linear action $B \to$ on vertices($\mathcal{P}$). Consider $A \in \mathfrak{I}(\mathcal{P})$ commuting with B.
Is $\lambda(\mathcal{P}) = \lambda^B(\mathcal{P})$?

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The work on Table Loci is joint with Mats Boij. We appreciate discussions with and helpful comments by Don King, Alfred Noel, George McNinch, and a conversation of Rui and Tony with Barry Mazur. We are grateful for the insights of P. Oblak, T. Košir and others who contributed questions and results that have been important to our work. We appreciate the notes of Rick Porter on LaTeX, xy-pic, and his advice.

Merci de votre attention and vos questions !

Appendix : $\mathfrak{Q}(P)$ and its smallest part (L.Khatami)

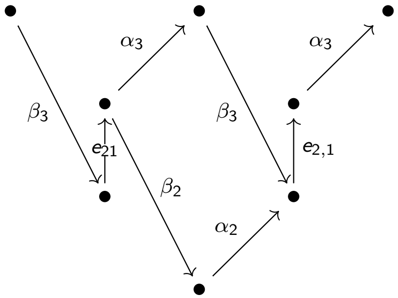


FIGURE : Diagram of the poset $\mathcal{D}_P$: $P = (3, 2, 2, 1)$.

Def. (U-chain)

A U-chain C_i in $\mathcal{D}_P$ is the saturated (maximal) chain through the union of three subsets of vertices :

(i) All rows of length $i, i-1$, corresponding to an AR subpartition of P.

(ii) A descending chain from the source – the top left vertex of $\mathcal{D}_P$ – to the vertex at the start of the lowest length-i row.

(iii) An ascending chain from the vertex at the end of the highest length-i row to the sink - the top right vertex of $\mathcal{D}_P$.

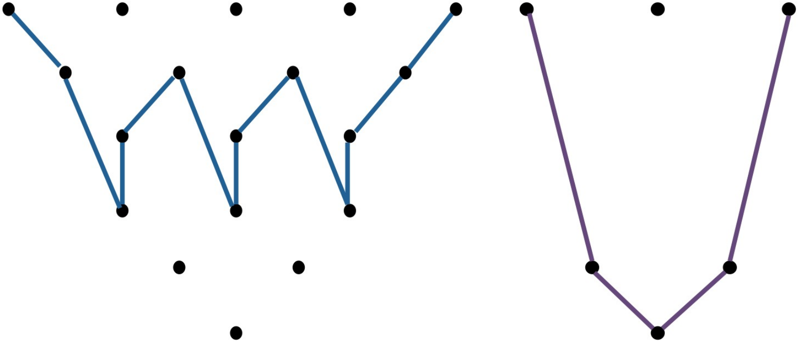


FIGURE : U-chain C_4 for $P = (5, 4, 3, 3, 2, 1)$ and new U-chain for $P' = (3, 2, 1)$.[Source : LK NU GASC talk 2013]

Oblak Recursive Conjecture

We obtain $\mathfrak{Q}(P)$ as follows from $\mathcal{D}_P$:

(i) Choose a maximum length U-chain in $\mathcal{D}_P$. Its length is q_1, the largest part of $\mathfrak{Q}(P)$.

(ii) Remove the vertices in the chain from $\mathcal{D}_P$, obtaining a smaller partition P'. If $P' \neq \emptyset$ then $\mathfrak{Q}(P) = (q_1, \mathfrak{Q}(P'))$ (go to (i).).

Warning. The poset $\mathcal{D}_{P'}$ in the Oblak recursion is *not* in general a subposet of $\mathcal{D}_P$.

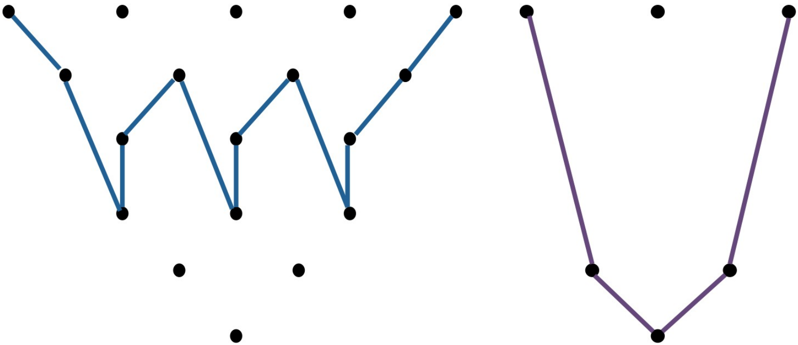


FIGURE : U-chain for $P = (5, 4, 3, 3, 2, 1)$ and new U-chain for $P' = (3, 2, 1)$. So $\mathfrak{Q}(P) = (12, 5, 1)$.

Theorem (P. Oblak[Obl1] – Index of $\mathfrak{Q}(P)$)

The index (largest part) of $\mathfrak{Q}(P)$ is the length of the longest U-chain in $\mathcal{D}_P$.

Theorem (L. Khatami [Kh1] – $Ob(P) = \lambda_U(\mathcal{D}_P)$)

The partition $Ob(P)$ obtained by the Oblak recursive process is independent of the choices of AR subpartitions ; and $Ob(P) = \lambda_U(\mathcal{D}_P)$, obtained as $\lambda(\mathcal{D}_P)$ below by restricting to sets of U-chains.

Definition

$P \geq P'$ in the orbit closure (Bruhat) order if $\forall i \ \sum_{u=1}^{i} p_u \geq \sum_{u=1}^{i} p'_u$.

Theorem (I,L.Khatami [IKh])

$\mathfrak{Q}(P) \geq Ob(P)$.

Proof idea. For each maximal-length set of s U-chains, we specify an $A \in \mathcal{N}_B$ such that $\dim_{\mathsf{k}} \mathsf{k}[A] \circ \{v_1, \ldots, v_s\}$ where the v_i are initial elements, agrees with the sum of the first s parts of $Ob(P)$. This involves an analysis of the sets of chains from the v_i to all the vertices covered by the s U-chains.

Def. (C. Greene et al, see[BrFo])

Let $\mathcal{D}$ be a poset without loops. Define $c_i = \max\#$ vertices covered by i chains. Set

$$\lambda(\mathcal{D}) = (c_1, c_2 - c_1, c_3 - c_2, \ldots).$$

Theorem (C. Greene, S. Poljak, E.R. Gansner, see [BrFo])

Let $\mathcal{D}$ be any finite poset without loops, and let A be a generic nilpotent matrix in the incidence algebra $\mathfrak{I}(\mathcal{D}_P)$. Then the Jordan type $P_A = \lambda(\mathcal{D})$.

Definition ([Kh1])

$\lambda_U(\mathcal{D}_P)$ is obtained by replacing arbitrary chains c_i in the definition of $\lambda(\mathcal{D}_P)$ by U-chains.

Let $P = ((p+s-1)^{n_s}, \ldots, p^{n_1})$ be an s-spread : $n_i > 0$ for $1 \leq i \leq s$. Set

$$\mu(P) = \min\{pn_{2i-1} + (p+1)n_{2j} | 1 \leq i \leq j \leq r_P\}$$

Note : For s odd $r_P = (s+1)/2$ so $n_{2r_P} = 0$ and $\mu(P) = p \cdot \min\{n_{2i-1} \mid 1 \leq i \leq r_P\}$.

Theorem (L.Khatami [Kh2])

For P a spread, $\mu(P)$ is the # of disjoint length-r_P antichains in $\mathcal{D}_P$.

Fact : [Gre, BrFo] $\lambda(\mathcal{D}_P)_{\min} = \#$ length r_P anti-chains in $\mathcal{D}_P$.

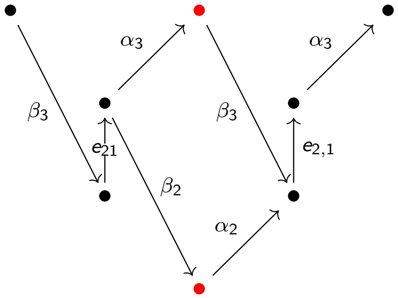


FIGURE : $\mu(P) = 1$ for $P = (3, 2, 2, 1)$, $Q(P) = (7, 1)$

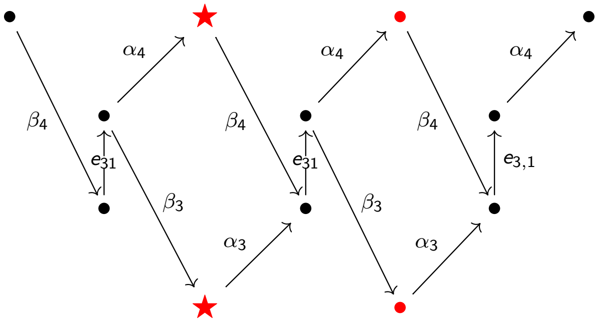


FIGURE : $\mu(P) = 2$ for $P = (4,3,3,2)$, $\mathfrak{Q}(P) = (10,2)$

Appendix 2

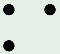
Combinatorial Relation between $\mathcal{T}(Q)$ and Durfee squares

Definition ($\mathcal{DH}(Q)$, Q stable)

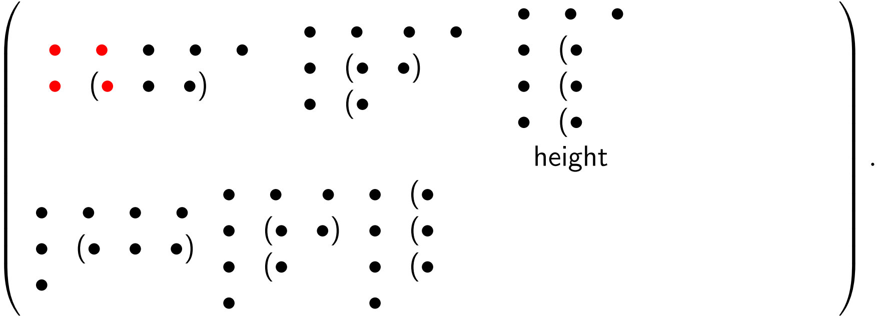
Let Q be a stable partition. Denote by $\mathcal{DH}(Q)$ the set of all partitions having diagonal hook lengths Q.

Example ($\mathcal{DH}(Q)$ for $Q = (6,3)$)

The inside diagonal hook h_{22} has length 3 so can be

$P' = (3)$  , $(2,1)$  , or $(1,1,1)$  .

Then a diagonal hook h_{11} of length 6 is folded around P' ; in each case there are two positions : adding one, or two parts to P'. So $\mathcal{DH}(Q), Q = (6,3)$ is



$$= \begin{pmatrix} (5,4) & (4,3,2) & (3,2,2,2) \\ (4,4,1) & (3,3,2,1) & (2,2,2,1) \end{pmatrix}.$$

Corollary (Bijection $\mathcal{T}(Q)$ et $\mathcal{DH}(Q)$.)

Let $Q = (u, u - r)$. There is a bijection $\theta : \mathcal{T}(Q) \to \mathcal{DH}(Q)$ that preserved the number of parts of P.

Proof. It is simple to write the tables $\mathcal{DH}(Q)$ by adding longer diagonal hooks ; so it is easy to count $|\mathcal{DH}(Q|)$. It's the same number for $\mathcal{T}(Q)$ after the Theorem. We take $\theta(\mathcal{T}_{ij}(Q)) = \mathcal{DH}_{i,j}(Q)$.

Question

Can we define θ^{-1} combinatorially ? A "jeu de taquin" ?

If we can extend the definition of θ to Q with $k > 2$ parts, this can help construct the tables $\mathcal{T}(Q)$, as $\mathcal{DH}(Q)$ is easy to write down.

Example (θ for $Q = (6,3)$)

The map $\theta(\mathcal{T}(Q)_{ij}) = \mathcal{DH}(Q)_{ij}$. Here

$$\mathcal{T}(Q) = \begin{pmatrix} (6,3) & (6,[3]^2) & (6,[3]^3) \\ (5,[4]^2) & (5,[4]^3) & (5,[4[^4) \end{pmatrix}.$$

$$\mathcal{DH}(Q) = \begin{pmatrix} (5,4) & (4,3,2) & (3,2,2,2) \\ (4,4,1) & (3,3,2,1) & (2,2,2,1) \end{pmatrix}.$$

Example (Case $r_P = 1$, $\mathrm{dh}(P)$ has 1×1 Durfee square.)

Let $n = 5$, $Q = (5)$.
$\mathcal{T}(Q) = \big([5], [5]^2, [5]^3, [5]^4, [5]^5\big)$
$DH(Q) = \big((5), (4,1), (3,1^2), (2,1^4), (1^5)\big)$ (single diagonal hook).

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