1. The topological space $X$ is *locally compact* if the following condition is satisfied: For every point $x \in X$, there is a compact subset $K \subseteq X$ that contains an (open) neighborhood of $x$.

Let $X$ be a locally compact Hausdorff space, and let $\{\infty\}$ be a single point disjoint from $X$. Define $Y = X \cup \{\infty\}$, and define a subset $U \subseteq Y$ to be open if:

- $U \subset X$ and $U$ is open in $X$, or
- $\infty \in U$ and $X \setminus U$ is compact in $X$.

(a) Show that $Y$ is a topological space.

(b) Show that $Y$ is compact and Hausdorff.

(c) What is $Y$ if $X = \mathbb{R}^n$?

2. Prove or disprove the following:
   (a) If $X$ and $Y$ are path-connected, then $X \times Y$ is path-connected.

   (b) If $A \subseteq X$ is path-connected, then $\overline{A}$ is path-connected.

   (c) If $X$ is locally path-connected, and $A \subseteq X$, then $A$ is locally path-connected.

   (d) If $X$ is path-connected, and $f : X \to Y$ is continuous, then $f(X)$ is path-connected.

   (e) If $X$ is locally path-connected, and $f : X \to Y$ is continuous, then $f(X)$ is locally path-connected.

3. Let $X = \{(z, y) \in \mathbb{C}^2 \mid y = z^3\}$, let $Y = X \setminus \{(0, 0)\}$, and let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

   (a) Show that the map $p : X \to \mathbb{C}$, $(z, y) \mapsto z$ is a homeomorphism.

   (b) Show that the map $q : Y \to \mathbb{C}^*$, $(z, y) \mapsto y$ is a covering map, and identify the fiber $F = q^{-1}(1)$ of this map. Is the cover a regular cover?

   (c) Fix a basepoint $x_0 \in F$, and determine the induced homomorphism $q_\ast : \pi_1(Y, x_0) \to \pi_1(\mathbb{C}^*, 1)$. How does $\pi_1(\mathbb{C}^*, 1)$ act on $F$?
4. Let $T^2$ be the 2-dimensional torus.
   (a) Identify (up to homeomorphism) all the path-connected spaces $E$ that appear as the total space of a covering map $p: E \to T^2$. Which one of those is the universal cover?
   (b) Prove, or give a counterexample to the following assertion: Every continuous map $S^1 \to T^2$ is null-homotopic.
   (c) Prove, or give a counterexample to the following assertion: Every continuous map $S^2 \to T^2$ is null-homotopic.

5. Let $M_g$ be a compact, connected, orientable surface of genus $g \geq 0$, and let $M_{g,r}$ be this surface, with $r \geq 1$ distinct points removed.
   (a) What is the fundamental group of $M_{g,r}$?
   (b) Compute the homology groups $H_i(M_{g,r}, \mathbb{Z})$, for all $i \geq 0$.
   (c) Let $N \to M_{g,r}$ be a $k$-fold cover. Show that if $N$ is connected then $N = M_{h,s}$, for some $h \geq 0$ and $s \geq 1$.
   (d) Find a relation among the integers $g, r, k, h, s$.

6. If $M$ and $N$ are two connected, oriented manifolds of dimension $n$, their connected sum, $M \sharp N$, is obtained by removing an open $n$-disk from each manifold, and identifying the boundaries of the two disks by an orientation-preserving homeomorphism.
   (a) Express the Euler characteristic $\chi(M \sharp N)$ in terms of $\chi(M)$ and $\chi(N)$.
   (b) Suppose $n > 2$. Express the fundamental group $\pi_1(M \sharp N)$ in terms of $\pi_1(M)$ and $\pi_1(N)$.