1. Let $\mathbb{R}_h$ denote the set $\mathbb{R}$ with the topology generated by all half-open intervals of the form $[a, b)$. Let $A \subset \mathbb{R}_h$.
(a) Show that $x$ is in the closure of $A$ if and only if there is a sequence $x_n \in A$ such that $x_n \geq x$ and $|x_n - x| \rightarrow 0$.
(b) Give an $(\epsilon, \delta)$-definition for what it means for a function $f : \mathbb{R}_h \rightarrow \mathbb{R}$ to be continuous.

2. Let $F_t : X \rightarrow X$, $t \in [0, 1]$ be a homotopy where $F_0 = F_1 = id_X$, the identity map of $X$. Let us fix an $x_0 \in X$ and define $\gamma(t) = F_t(x_0) : [0, 1] \rightarrow X$. Then $\gamma$ represents an element $[\gamma] \in \pi_1(X,x_0)$. Show that $[\gamma]$ commutes with every element of $\pi_1(X,x_0)$.

3. Determine the number of maps $f : S^2 \rightarrow S^1$ up to homotopy.

4. Let $X$ be the unit sphere in $\mathbb{R}^3$ together with the segment $[(1,0,0);(-1,0,0)]$. Let $Y$ be a torus together with a disk that “covers the hole in the donut”.
(a) Define what is a homotopy equivalence between spaces.
(b) Show that $X$ and $Y$ are homotopy equivalent. (Hint: find a third space they are both equivalent to!)

5. Let $f : S^1 \rightarrow S^1$ be the rotation by 120 degrees. Let $X$ be the space obtained from the unit disk (whose boundary is $S^1$) by gluing every point $x$ on the boundary to $f(x)$ (and then also to $f(f(x)))$.
(a) Find a connected space $Y$ together with a 3-sheeted covering map $f : Y \rightarrow X$.
(b) What is the fundamental group of $X$?

6. (a) Let $X,Y$ be finite CW-complexes and $f : X \rightarrow Y$ is a covering map with $k$ sheets. Find a relation between $\chi(X), \chi(Y)$, and $k$.
(b) Let $F_p$ denote the orientable surface of genus $p$. Determine for which pairs $(p,q)$ does there exist a covering map $f : F_p \rightarrow F_q$. 

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Qualifying Exam in Topology

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Do the following eight problems. Give proofs or justifications for each statement you make. Draw pictures when needed. Be as clear and concise as possible. Show all your work.
7. State the Seifert - van Kampen theorem. Let $X$ be the space obtained by taking the solid cube $[0, 3] \times [0, 3] \times [0, 3]$, and removing from it the interiors of all $[i, i + 1] \times [j, j + 1] \times [k, k + 1]$ little cubes. Compute the fundamental group of $X$ using the Seifert - van Kampen theorem.

8. Let $T$ be the (2-dimensional) torus, and let $D \subset T$ be a small open disk. Let $X = T \setminus D$, and denote by $Y \subset X$ the boundary circle of $D$.

(a) Compute the fundamental group of $X$.

(b) Which element of this group does $Y$ represent?

(c) Is there a retraction $r : X \rightarrow Y$? (That is, a continuous map which is the identity when restricted to $Y$.)