1. Given an action $\Phi : G \to \text{Diff}(M)$ of a Lie group $G$ on a manifold $M$, set

$$\xi_M(p) = \frac{d}{dt} \bigg|_{t=0} \Phi(\exp(-t\xi)), \quad \xi \in \mathfrak{g} = \text{Lie}(G).$$

Show that $\xi \mapsto \xi_M$ is an action of the Lie algebra $\mathfrak{g}$ on $M$, i.e. for any two elements $\xi, \eta \in \mathfrak{g}$ the following equality holds

$$[\xi_M, \eta_M] = [\xi, \eta]_M.$$

(Here in the left hand side $[\cdot, \cdot]$ denotes the commutator of vector fields on $M$ and in the right hand side $[\cdot, \cdot]$ denotes the commutator in the Lie algebra $\mathfrak{g}$).

2. a. Formulate the Frobenius theorem.

b. On $\mathbb{R}^4$ endowed with the coordinate system $(x, y, z, t)$ we consider three vector field

$$X = t \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} - x \frac{\partial}{\partial t},$$

$$Y = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} - t \frac{\partial}{\partial z} + z \frac{\partial}{\partial t},$$

$$Z = -z \frac{\partial}{\partial x} + t \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} - y \frac{\partial}{\partial t}.$$

Is this distribution integrable? Explain.

3. Let $M = \mathbb{R}^3$ and let $E = M \times \mathbb{R}^2$ be a trivial vector bundle over $M$. Then the space of sections of $E$ is identified with the space of smooth vector valued functions $s : \mathbb{R}^3 \to \mathbb{R}^2$. Consider the connection on $E$ whose covariant derivative $\nabla : \Gamma(E) = C^\infty(\mathbb{R}^3) \to \Omega^1(M, E) = \Omega^1(\mathbb{R}^3, \mathbb{R}^2)$ is given by the formula

$$\nabla s = ds + A_1 s dx^1 + A_2 s x^1 dx^2 + A_3 s dx^3,$$

where

$$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

Let $\alpha = A_1 dx^1 + A_2 dx^2 + A_3 dx^3 \in \Omega^1(\mathbb{R}^3, \text{End}(\mathbb{R}^2))$.

(a) Define the notion of the curvature of the connection $\nabla$.

(b) Compute the curvature of this connection.
4. a. Give a definition of the Levi-Civita connection on a Riemannian manifold $M$.

b. Let $M \subset \mathbb{R}^n$ be a submanifold and let $g$ denote the Riemannian metric on $M$ induced by the standard scalar product on $\mathbb{R}^n$. Let $\nabla$ denote the connection on $TM$ defined as follows: for $x \in M$ and a vector field $X$ let $\gamma_X(t)$ be the integral curve of $X$ with $\gamma_X(0) = x$. Thus
\[
\frac{d}{dt} \gamma_x(t) = X(\gamma(t)).
\]

Then
\[
\nabla_X Y(x) := \Pi_x \frac{d}{dt} Y(\gamma_X(t)),
\]
where $\Pi_x : \mathbb{R}^n \rightarrow T_x M$ is the orthogonal projection. Show that $\nabla$ is the Levi-Civita connection.

5. a. Define the torsion $T(X, Y)$ of a connection $\nabla$ on $TM$. Show that it is $C^\infty$-linear in $X$ and $Y$, i.e. that for any smooth function $f \in C^\infty(M)$
\[
T(fX, Y) = T(X, fY) = fT(X, Y).
\]

b. Define Christoffel symbols of a connection on a Riemannian manifold.

c. Express the torsion in terms of the Christoffel symbols of the connection. (Please, show your work).