1. Let \( f \in L^1(X, \mu) \). Prove that for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that

\[
\left| \int_A f \, d\mu \right| < \epsilon
\]

whenever \( A \) is a measurable subset of \( X \) with \( \mu(A) < \delta \).

2. (a) Formulate the dominated convergence theorem.

(b) Assume \( f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is such that \( x \mapsto f(x, t) \) is measurable for each \( t \in \mathbb{R} \) and \( t \mapsto f(x, t) \) is continuous for each \( x \in \mathbb{R} \). Assume also that there is an integrable \( g : \mathbb{R} \rightarrow \mathbb{R} \) with \( |f(x, t)| \leq g(x) \) for each \( x, t \in \mathbb{R} \). Show that the function \( f^{[t]}(x) := f(x, t) \) is integrable for each \( t \) and the function

\[
F(t) := \int_{\mathbb{R}} f^{[t]}(x) \, dx
\]

is continuous.

(c) Show that the function

\[
F(t) := \int_0^\infty e^{-x} \cos(\pi t) \, dx
\]

is continuous.

3. Let \( 1 \leq q < p \leq \infty \).

(a) Show that \( L^p([0, 1]) \subset L^q([0, 1]) \).

(b) Show (by example) that the inclusion in (a) is strict.

(c) Give an example of a measure space \( (X, \mu) \) for which the inclusion \( L^p([0, 1]) \supset L^q([0, 1]) \) holds.
4. Let $f, g \in L^2(\mathbb{R})$. A convolution $f \ast g$ is defined by

$$ f \ast g(x) := \int_{\mathbb{R}} f(t)g(x - t) \, dt. $$

(a) Let $\hat{f}, \hat{g}$ denote the Fourier transforms of $f$ and $g$ respectfully. Find the Fourier transform of $f \ast g$.

(b) Let

$$ h(t) = \begin{cases} 1, & \text{for } |t| \leq 1/2; \\ 0, & \text{for } |t| > 1/2; \end{cases} $$

Compute $H := h \ast h$.

(c) Find the Fourier transform of $H$.

(d) Compute

$$ \int_{\mathbb{R}} \left( \frac{\sin x}{x} \right)^4 \, dx $$

*Hint:* Use Plancherel's theorem and part (c) of the problem.

5. Let $\omega$ be a 2-form on $\mathbb{R}^2 \setminus \{0\}$ defined by

$$ \omega = \frac{xdy - ydx}{x^2 + y^2}. $$

(a) Compute $d\omega$.

(b) Let $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the circle. Compute $\int_S \omega$.

(c) Is the form $\omega$ exact? Explain your answer.