1. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$-smooth function on $\mathbb{R}$ such that there exist constants $C > 0$ and $R > 0$ so that for any $k \geq 0$ and for any $x \in \mathbb{R}$,

$$|f^{(k)}(x)| \leq Ck! / R^k.$$ 

Show that for any $x_0 \in \mathbb{R}$ for any $x \in (x_0 - R, x_0 + R)$,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(x_0)}{k!} (x - x_0)^k,$$

and the series converges uniformly on any compact subset $K \subseteq (x_0 - R, x_0 + R)$.

(b) Assume that $f$ is as in (a). Is it true that $f(x) = \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} x^k$ all $x \in \mathbb{R}$? If your answer is ‘no’ give a counter example.

2. Let $f : [a, b] \to \mathbb{R}$, $a < b$, be bounded and Riemann integrable on $[a, b]$. Show that for any $k \geq 2$ the $k$-th power of $f$, $g(x) := (f(x))^k$, is also Riemann integrable on $[a, b]$. (Do not use general theorems about Riemann integral for proving this result. The result must be proved using the definition of the integral.)

3. Let $(X, \rho)$ be a metric space. Two sets $A, B \subseteq X$ are called separated if $\overline{A} \cap B = \emptyset$ and $\overline{B} \cap A = \emptyset$. Show that $A$ and $B$ are separated if and only if there exist open sets $U$ and $V$ in $X$ so that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

4. Consider the differential form $\kappa := p_1 dq_1 + p_2 dq_2$ in $\mathbb{R}^5 = \{(p_1, p_2, q_1, q_2, t)\}$. Find the absolute value of the integral $\int_S d\kappa \wedge dt$ where $S$ is the 3-dimensional surface in $\mathbb{R}^5$ defined by the equations $p_1^2 + q_1^2 = t^2$, $p_2^2 + q_2^2 = t^2$, and the inequality $0 \leq t \leq 1$.

5. Two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ in a vector space $X$ are called equivalent if there exist positive real constants $C > c > 0$ such that $\forall x \in X$, $c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1$. Show that any two norms in $\mathbb{R}^n$ are equivalent.