Linear Algebra

1. Let \( V \) be the subspace of \( \text{Mat}_{2 \times 2}(\mathbb{C}) \) consisting of trace 0 matrices

\[
V = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 0 \right\},
\]

and let

\[
E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Define a linear map \( f : \text{Mat}_{2 \times 2}(\mathbb{C}) \to \text{Mat}_{2 \times 2}(\mathbb{C}) \) by the formula

\[
f(X) = EX -XE.
\]

(a) Show that \( f \) restricts to a map \( f|_V : V \to V \), and give the matrix associated to \( f|_V \) (making sure to clearly state the relevant basis).

(b) Find the Jordan canonical form of \( f \).

2. Let \( V \) be an infinite dimensional vector space over a field \( K \), and \( \{v_i\}_{i \in I} \) be a basis of \( V \). For each \( i \in I \), let \( f_i : V \to K \) be defined by

\[
f_i(v_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}
\]

Prove that \( \{f_i\}_{i \in I} \) is linearly independent but does not span the dual space \( V^* \).

3. Let \( V \) be a finite dimensional vector space, and \( S_r(V) \) the \( r \)-th symmetric power of \( V \). Use universal properties to prove that there is a canonical isomorphism

\[
S_r(V^*) \cong S_r(V)^*
\]

of vector spaces. \textit{Your isomorphism should not involve the choice of a basis.}
4. Let $M$ be an $n \times n$ matrix with entries in $\mathbb{Z}$ and $\det(M) \neq 0$. Prove that all entries of $M^{-1}$ are in $\mathbb{Z}$ if and only if $\det(M) = \pm 1$.

5. Let $q: \mathbb{R}^3 \to \mathbb{R}$ be the quadratic form given by

$$q(x, y, z) = -x^2 + 4xy + 4xz + 2y^2 - yz + 2z^2$$

(a) Find the matrix $B$ associated to $q$.
(b) Reduce $B$ to diagonal form. (Hint: the eigenvalues are small integers.)
(c) State the signature of $q$.

**Groups**

Reminder: for answers where you are asked to calculate something, the calculations should be in the context of a proof which justifies your work in order to receive full credit.

6. Determine the number of isomorphism classes of abelian groups of order 360.

7. Let $G$ be the group of $3 \times 3$ matrices over $\mathbb{Z}/2\mathbb{Z}$ of determinant 1. Determine the order of $G$.

8. Let $G$ be a group and $Z$ the center of $G$, and suppose that $G/Z$ is cyclic. Prove that $G$ is abelian.

9. Let $G$ and $H$ be the abelian groups

$$G = \mathbb{Z}/30\mathbb{Z} \oplus \mathbb{Z}, \quad H = \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}.$$  

Determine the number of group homomorphisms from $G$ to $H$, that is, the number of elements of $\text{Hom}_{\mathbb{Z}}(G, H)$.

10. Determine the number of conjugates in $S_7$ of the permutation

$$\sigma = (2, 3, 1, 5, 6, 4, 7).$$

We use the notation $\sigma = (\sigma(1), \sigma(2), \sigma(3), \ldots, \sigma(7))$. For example: $\sigma(1) = 2, \sigma(2) = 3, \sigma(5) = 6, \text{ and } \sigma(7) = 7$. 

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